[Collatz in Dafny]

Hardware & Software Verification

John Wickerson & Pete Harrod

Lecture 10: SAT and SMT solving

Automatic proof

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  - e.g. in Dafny, to show that `invariant P` is preserved,
  - e.g. in Isabelle methods like `by auto`.
- How do these automatic provers work?

SAT queries

- Simple case: proofs about Boolean statements.

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  - $f = ((A \wedge \neg B \wedge C) \Longrightarrow (C \vee (B \wedge A)))$

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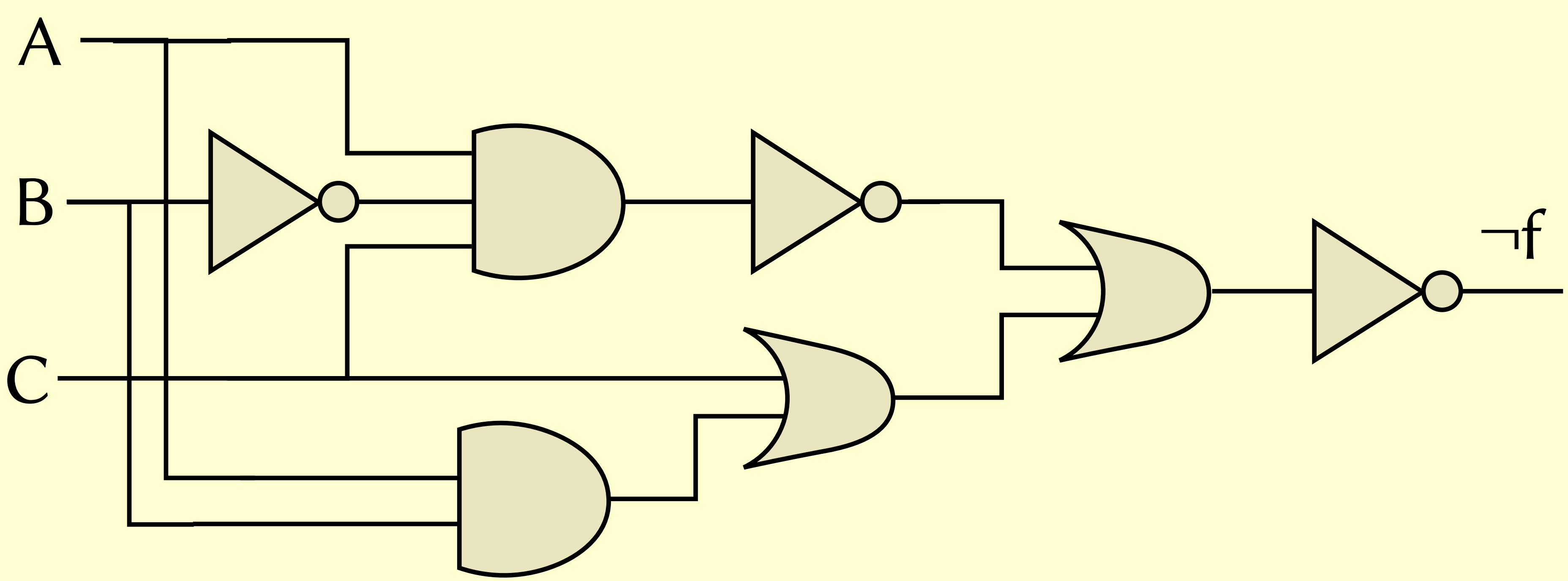

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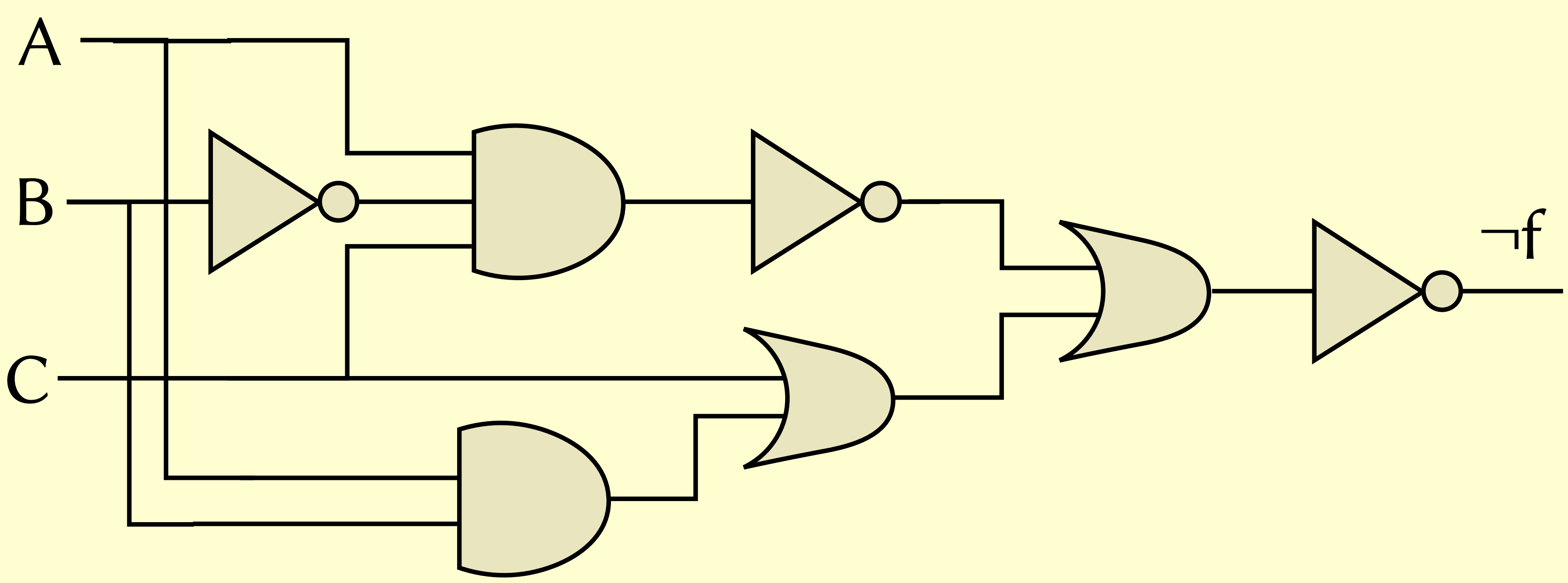

A	B	C	¬f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

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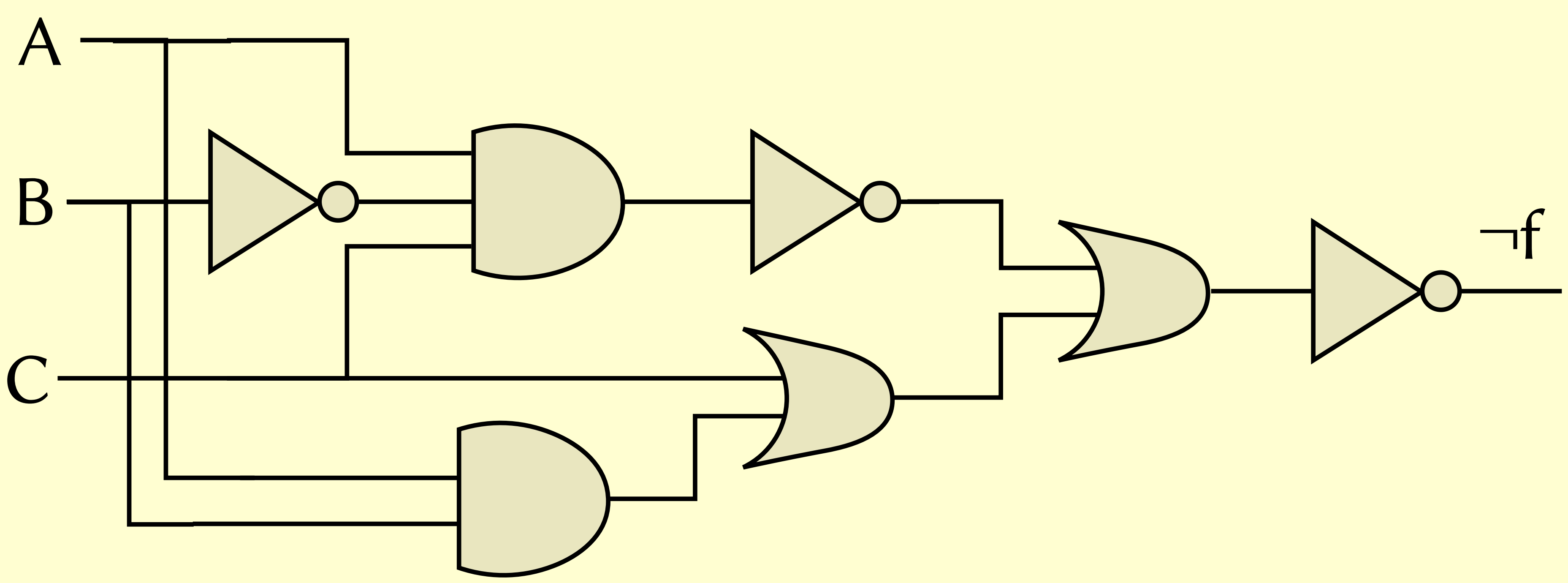

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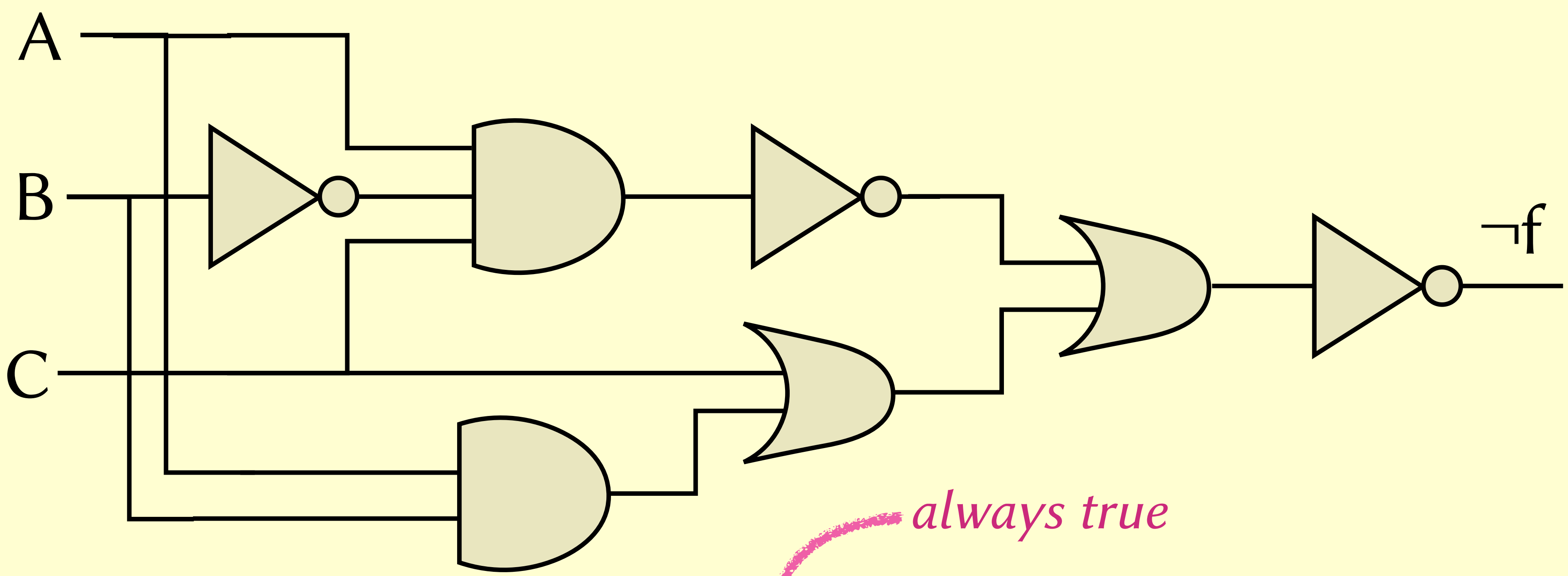

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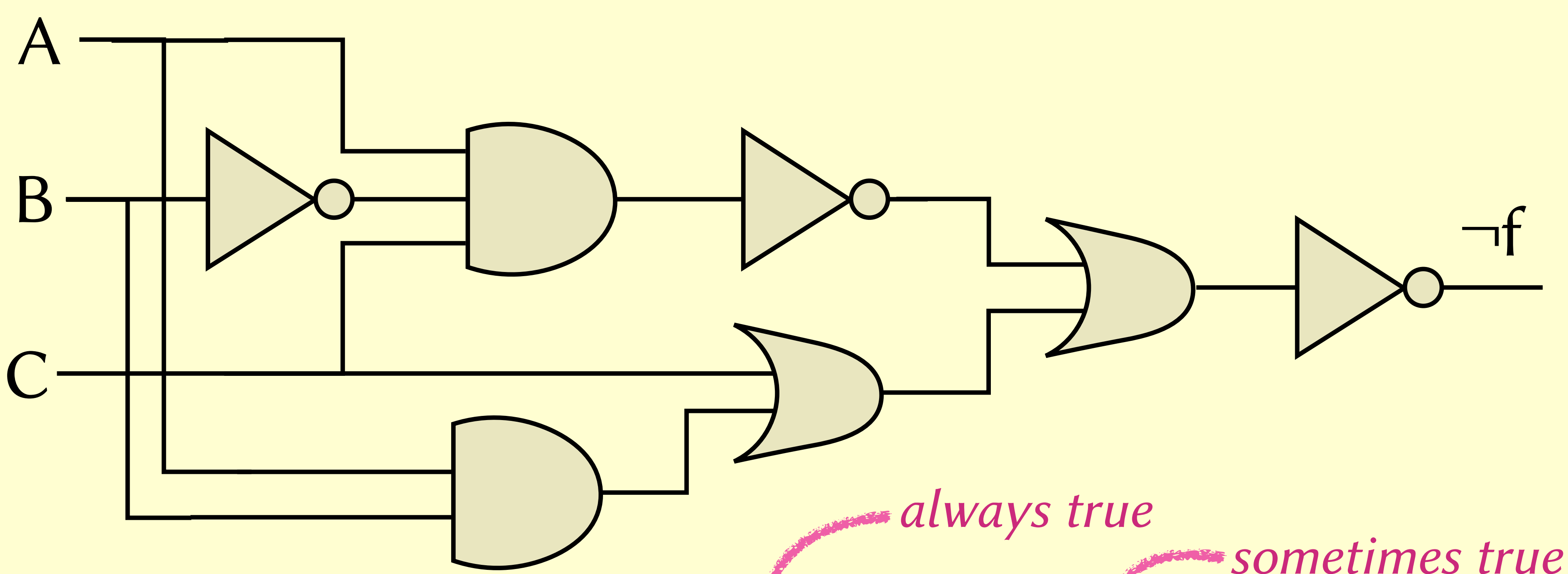

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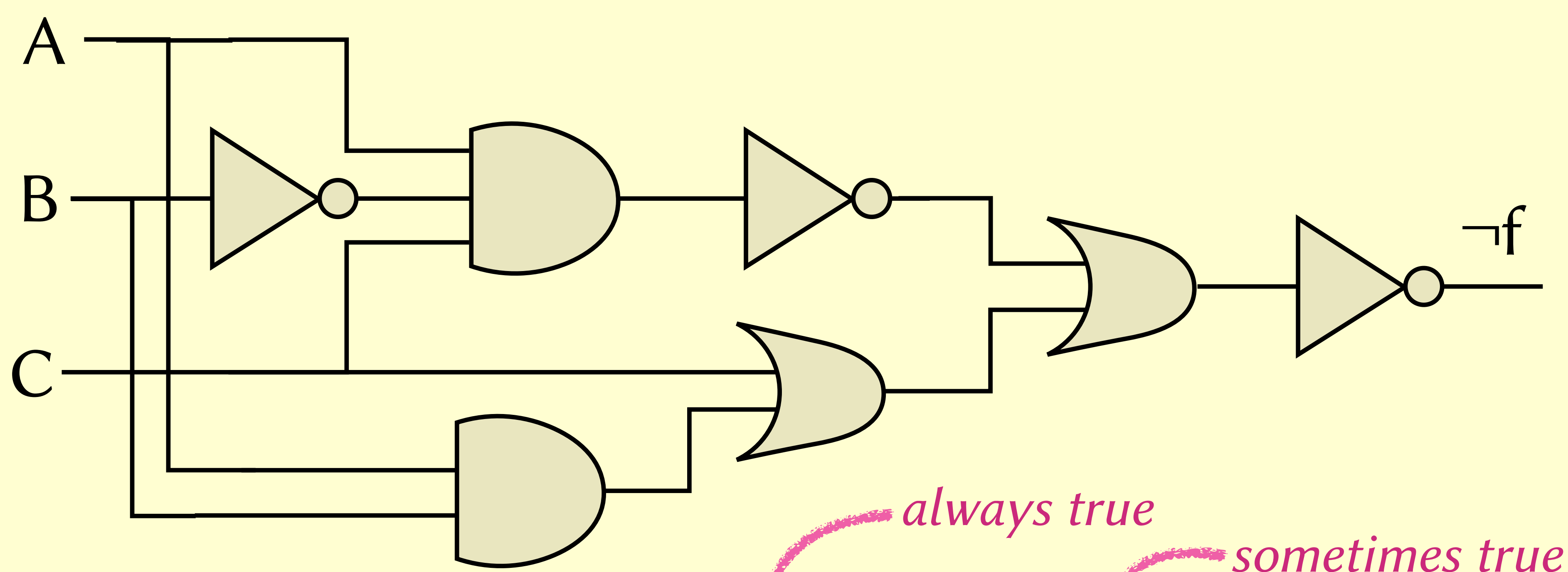

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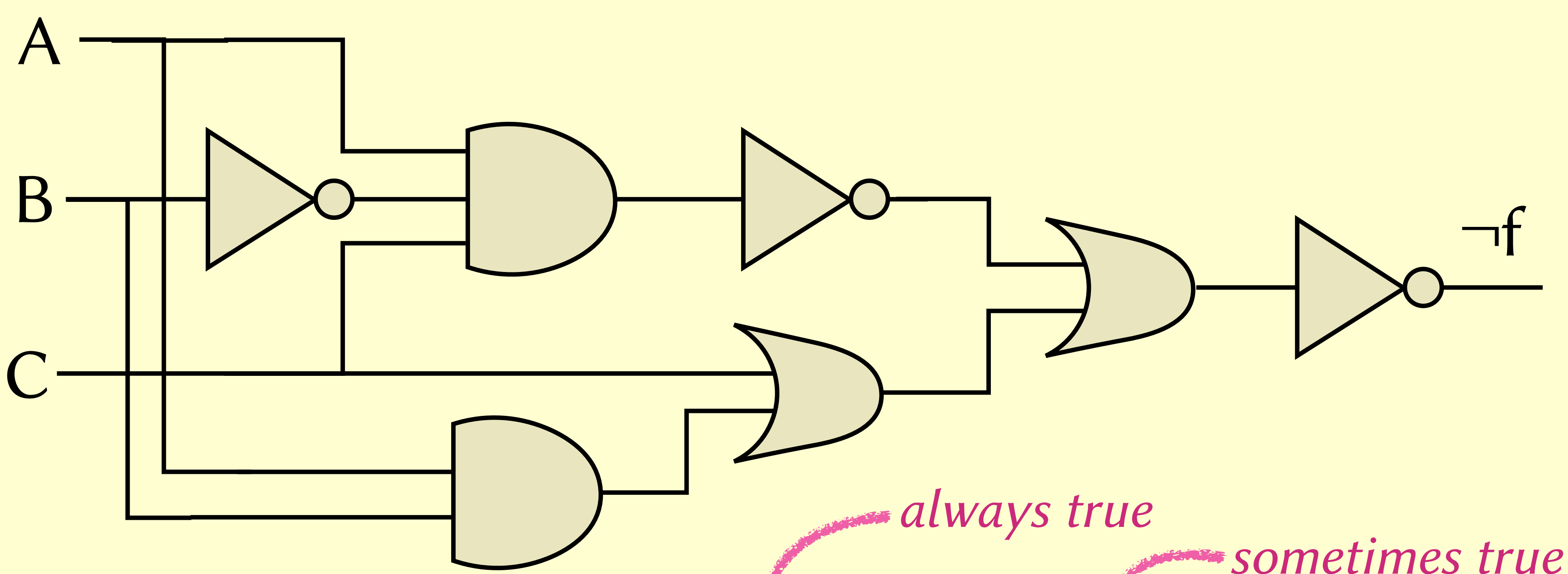

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always false *sometimes false*

SAT solving

- A simple algorithm:

```
for A in {0, 1}:
  for B in {0, 1}:
    for C in {0, 1}:
      if f(A,B,C) = 1:
        return ("SAT", [A, B, C])
return ("UNSAT")
```

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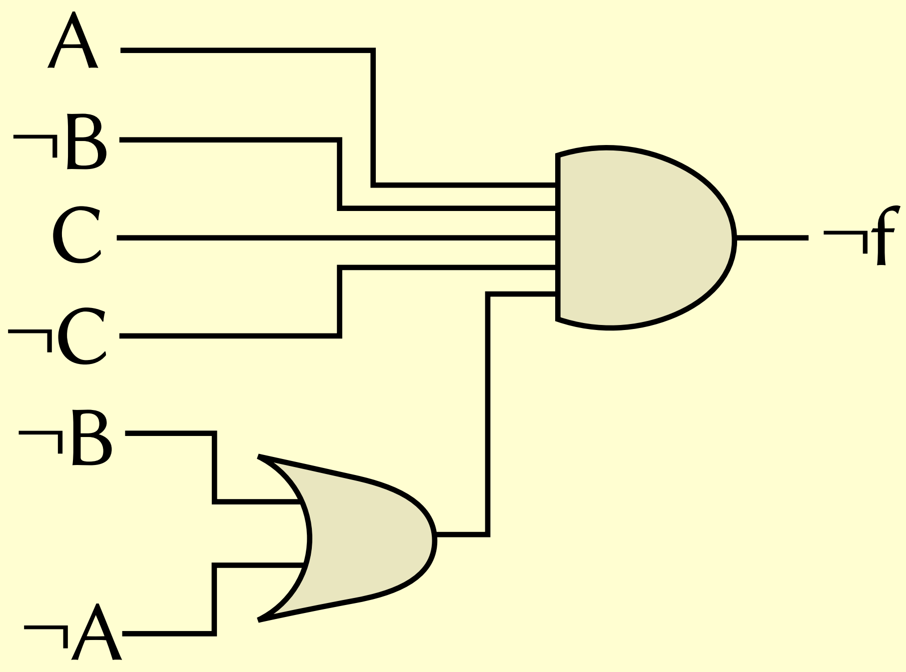

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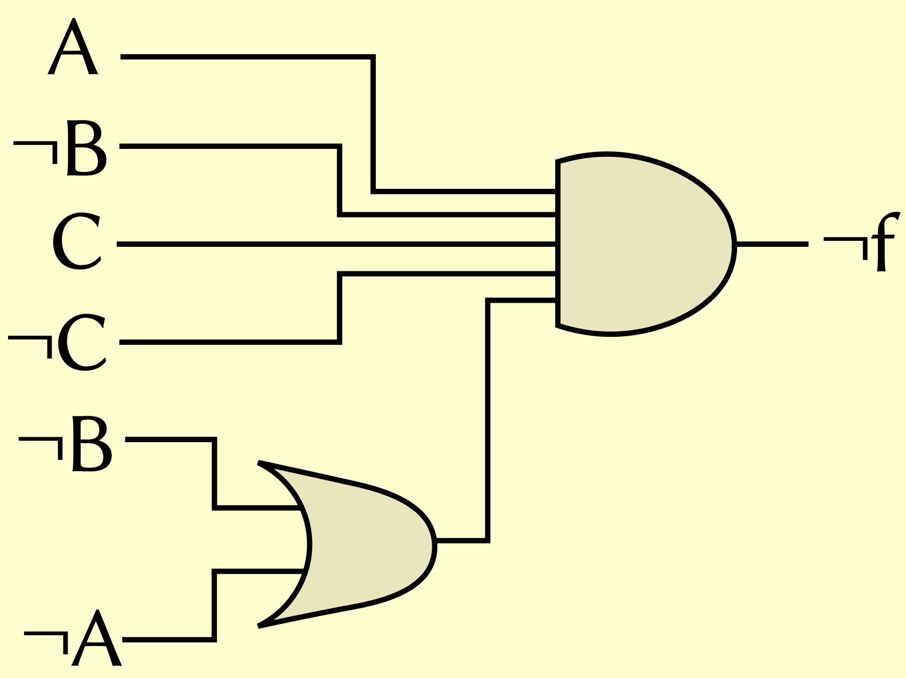

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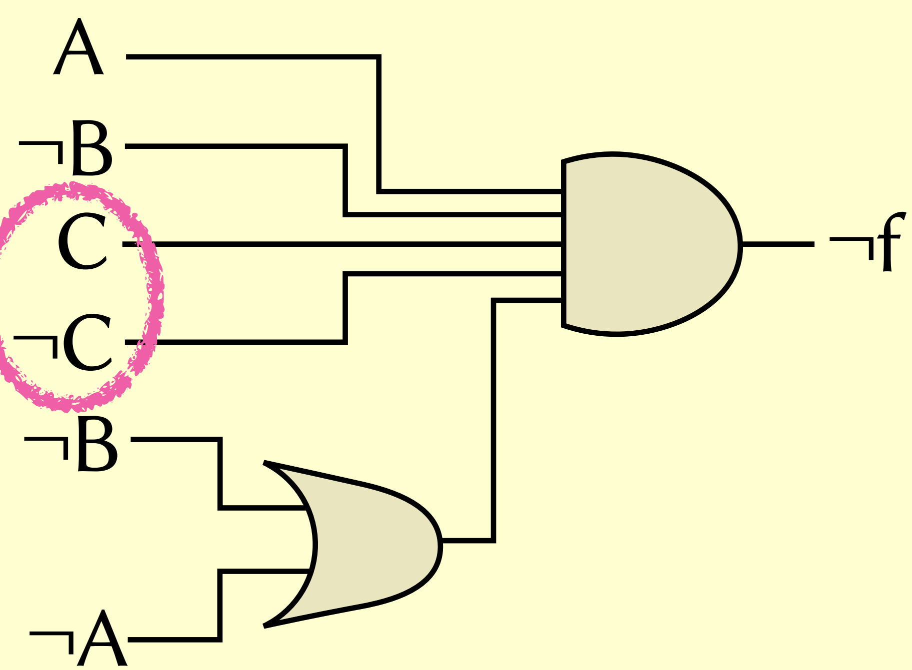

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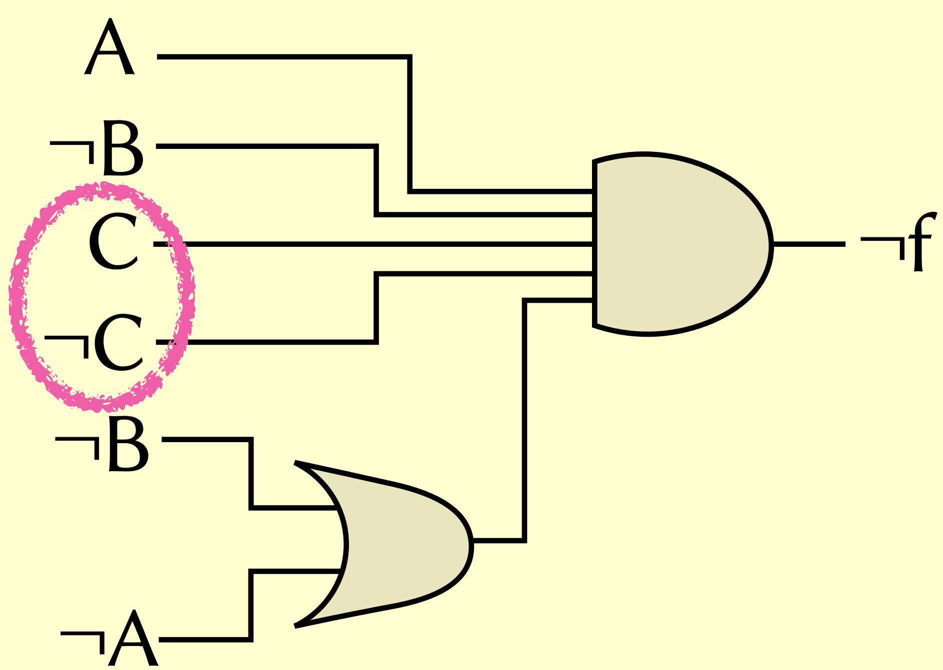

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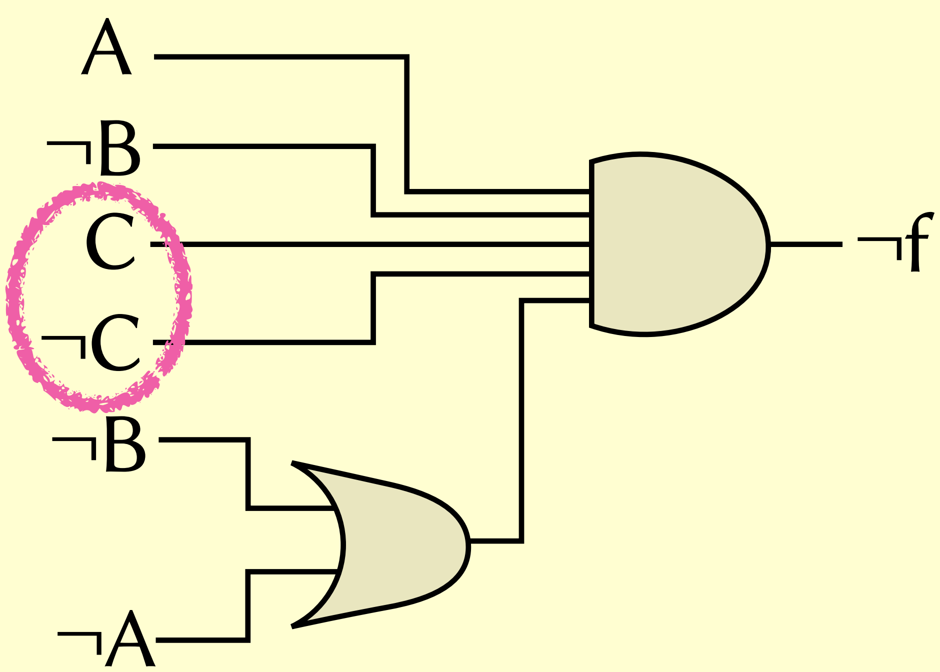

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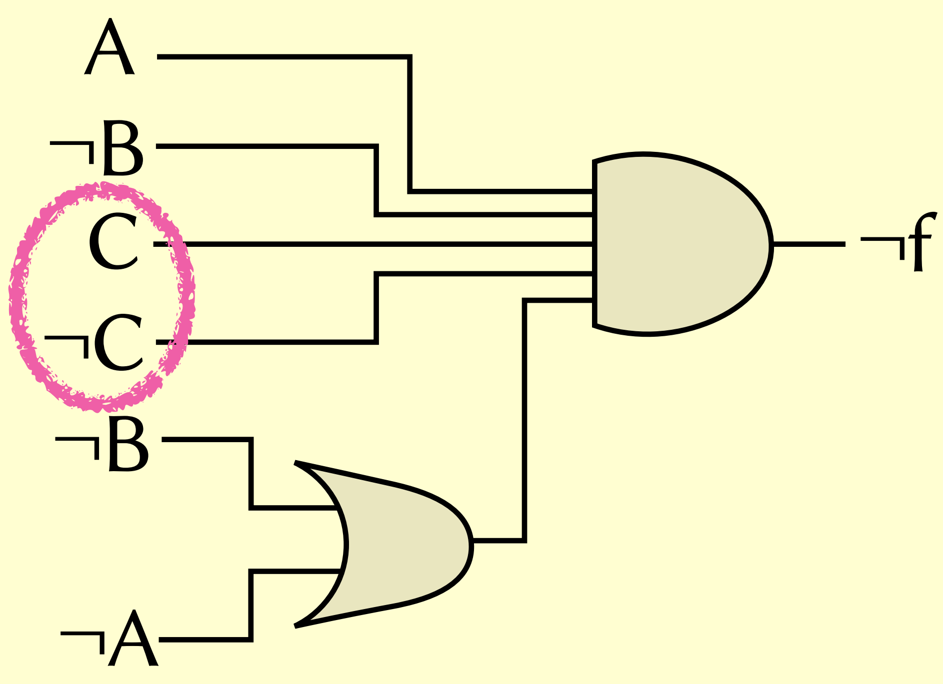

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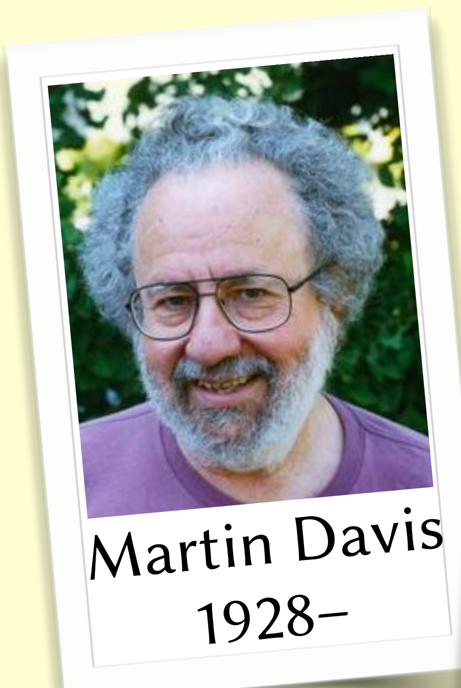
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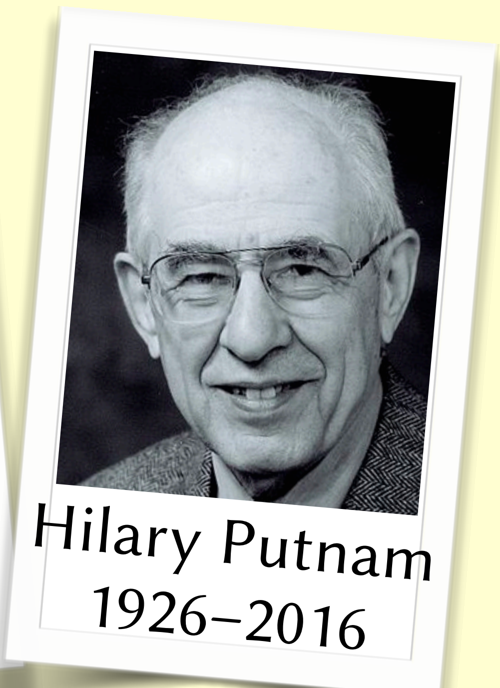
Hilary Putnam
1926–2016

The DP method

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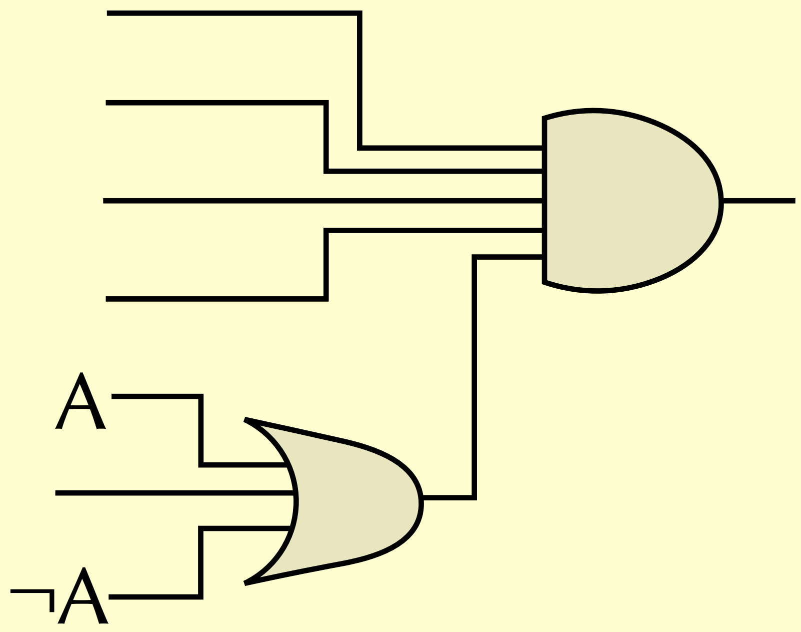

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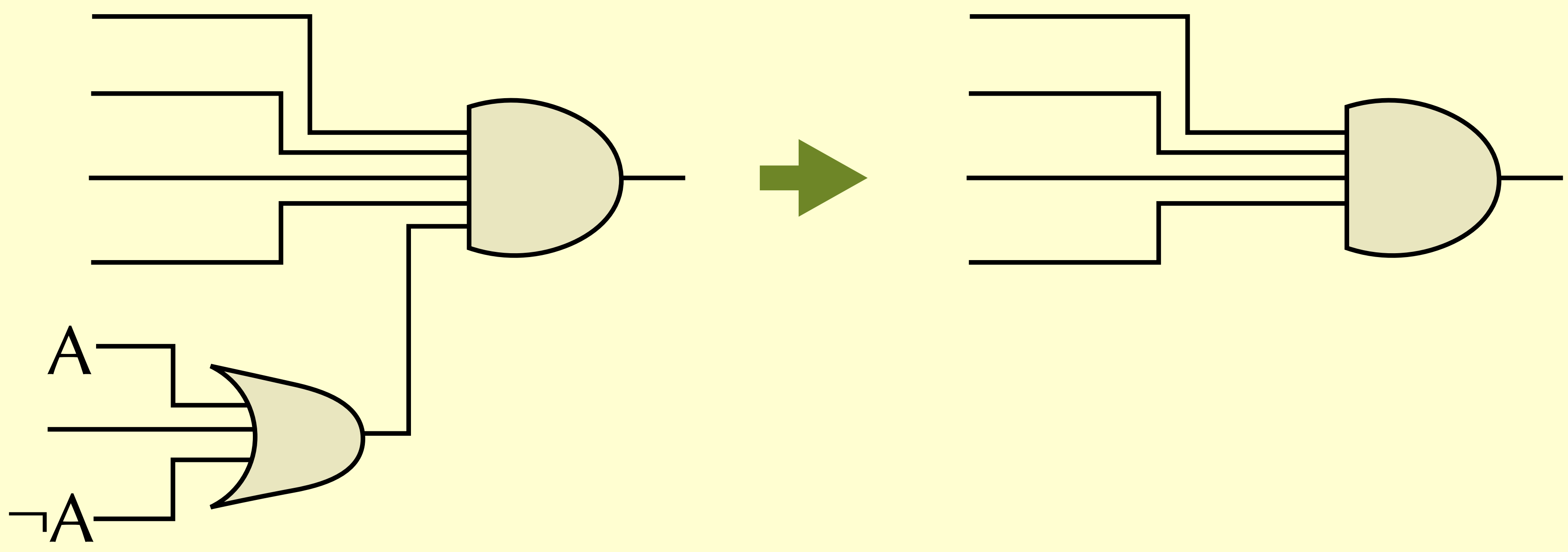

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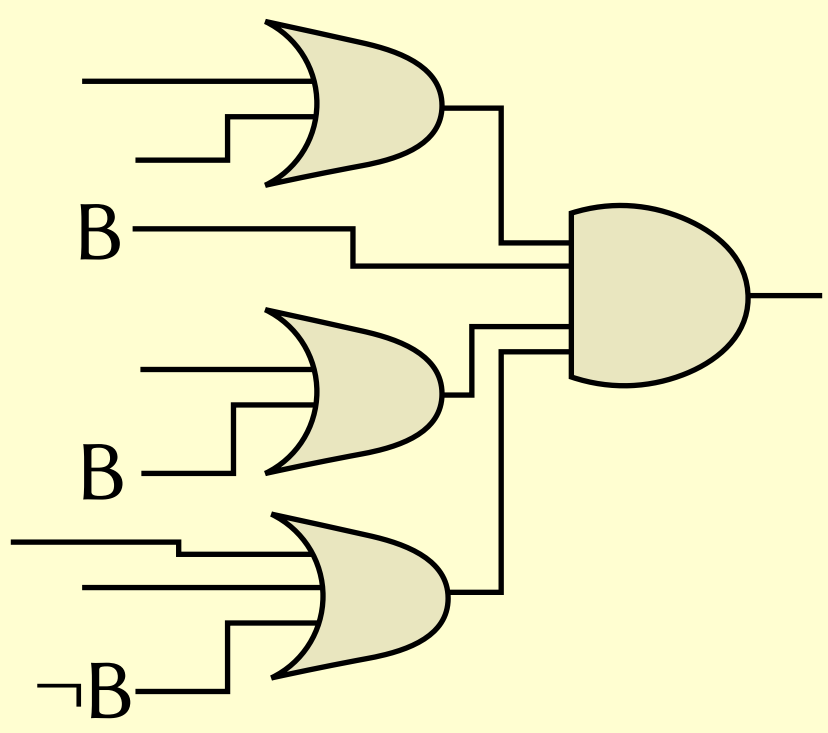
(The solution, if it exists, will surely involve setting L=1.)



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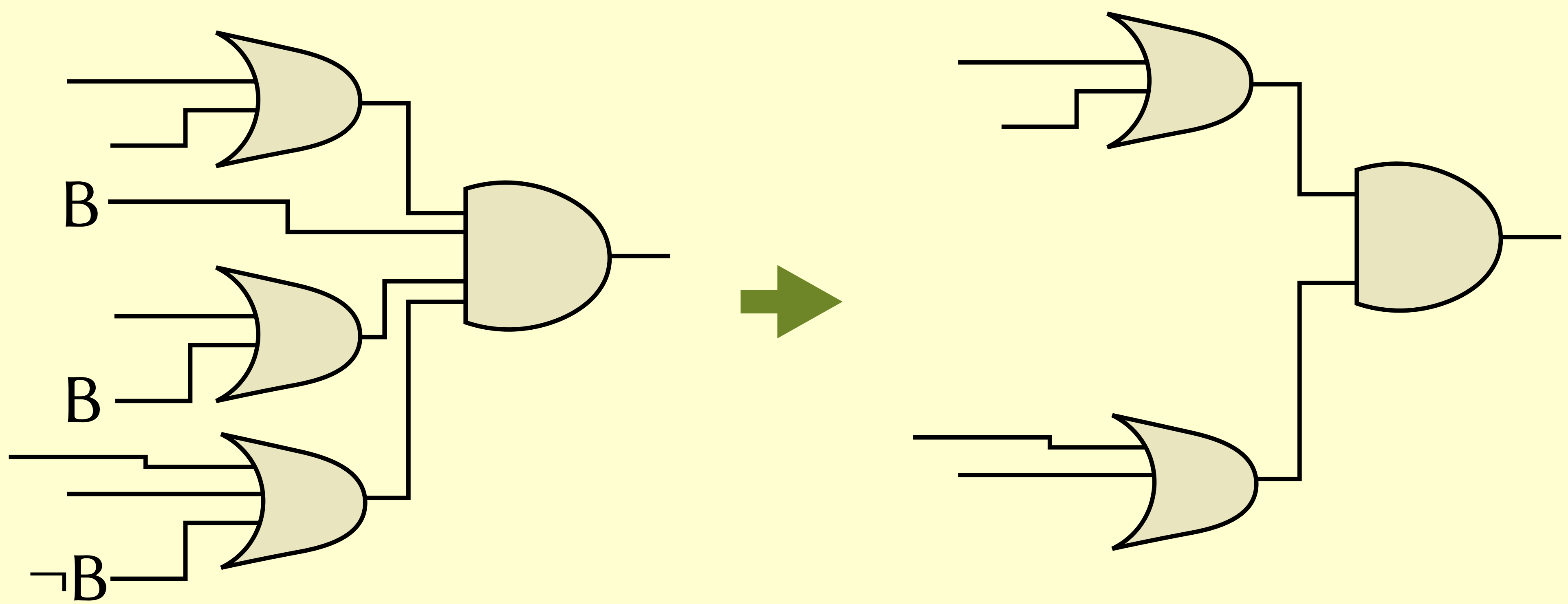

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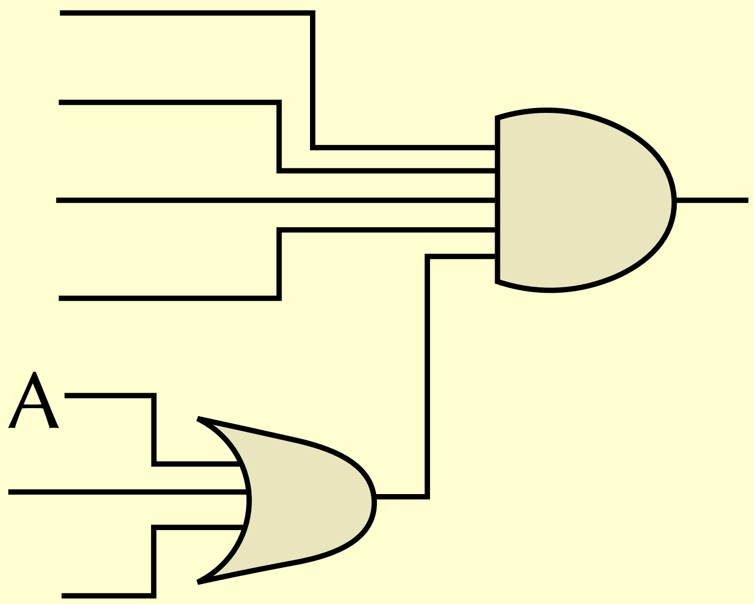

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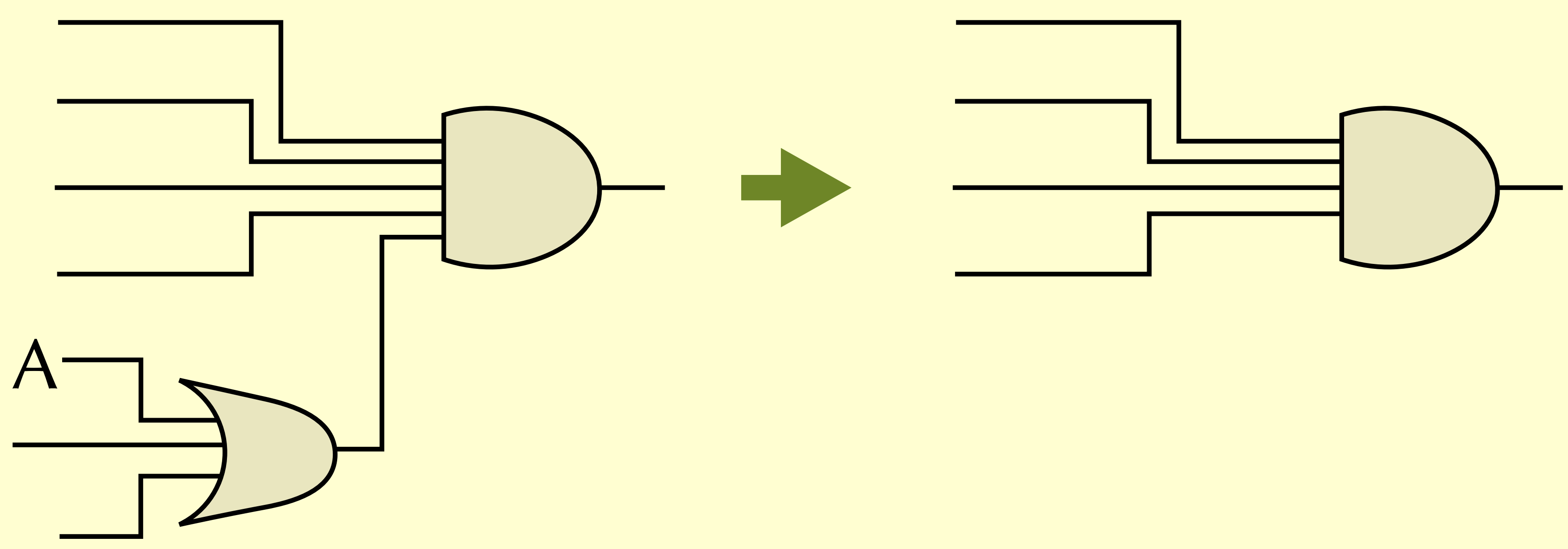

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6. Pick a literal L and repeat the above for the cases L=0 and L=1.

DP example 1

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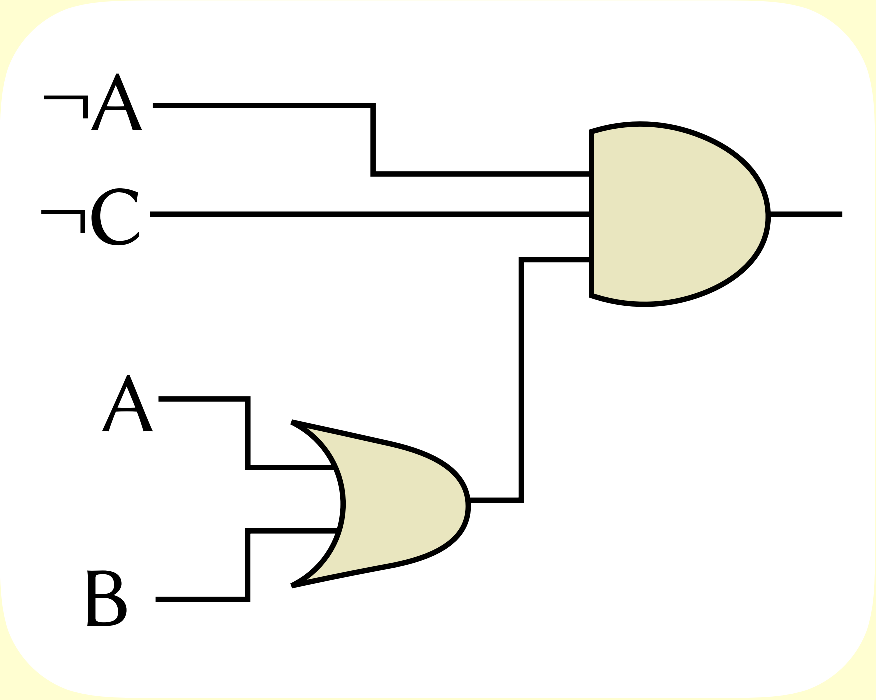

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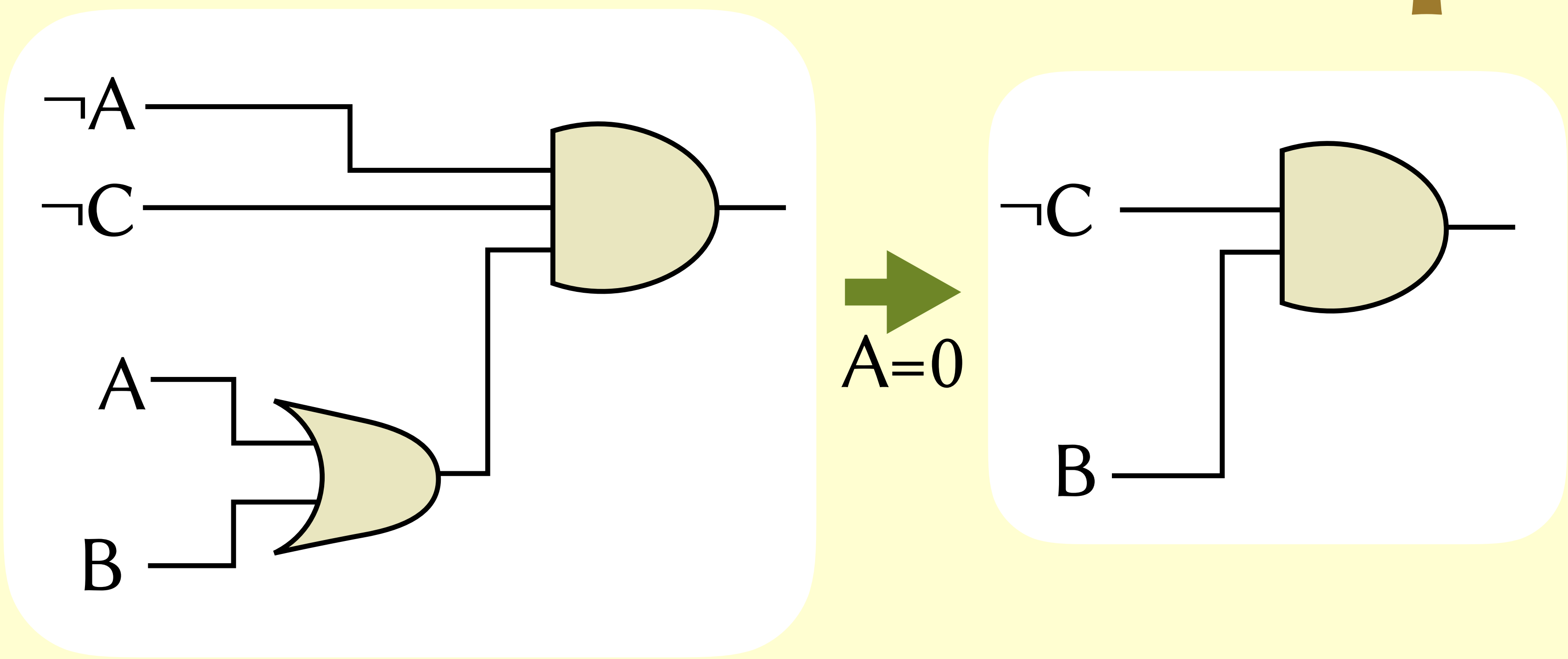

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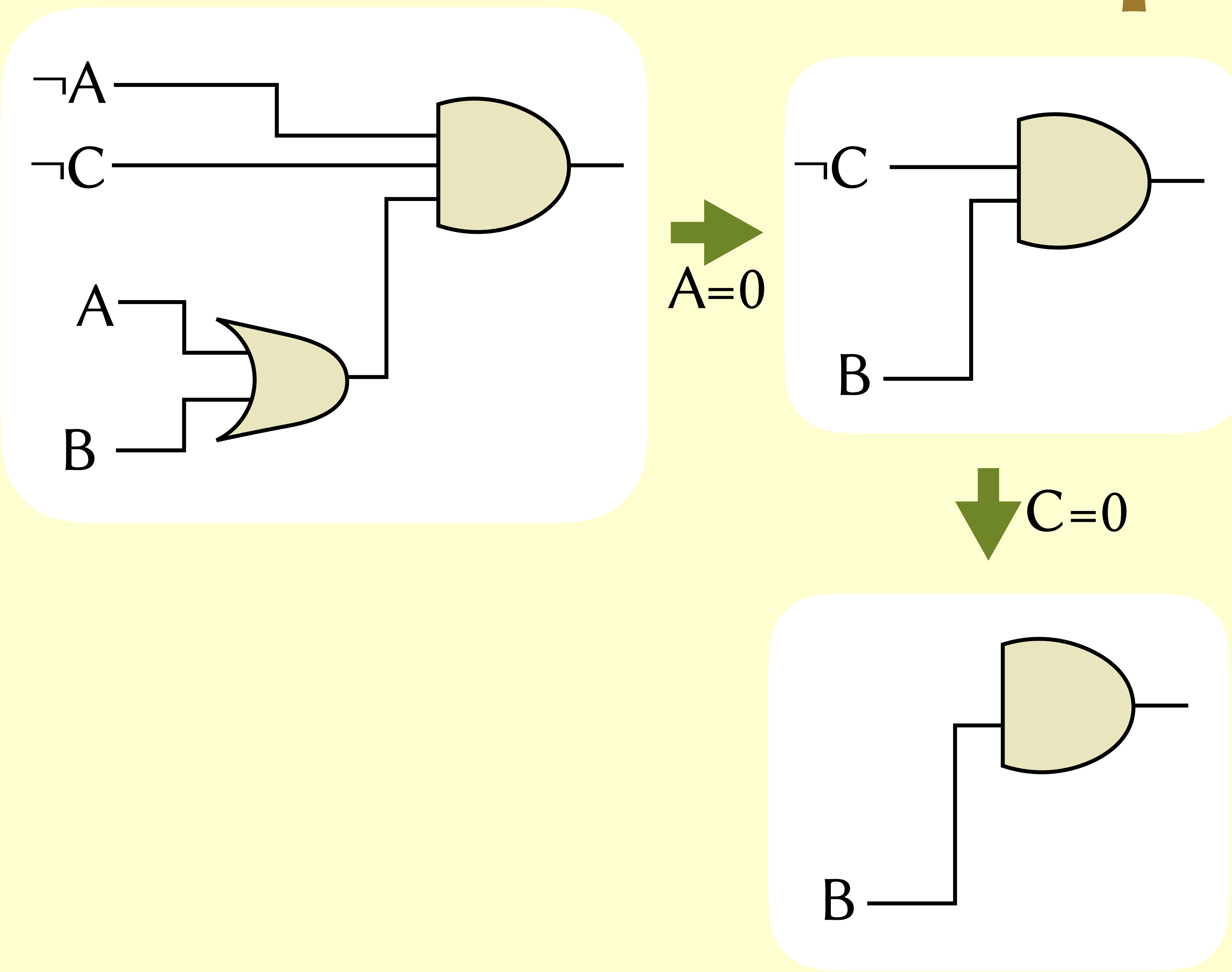

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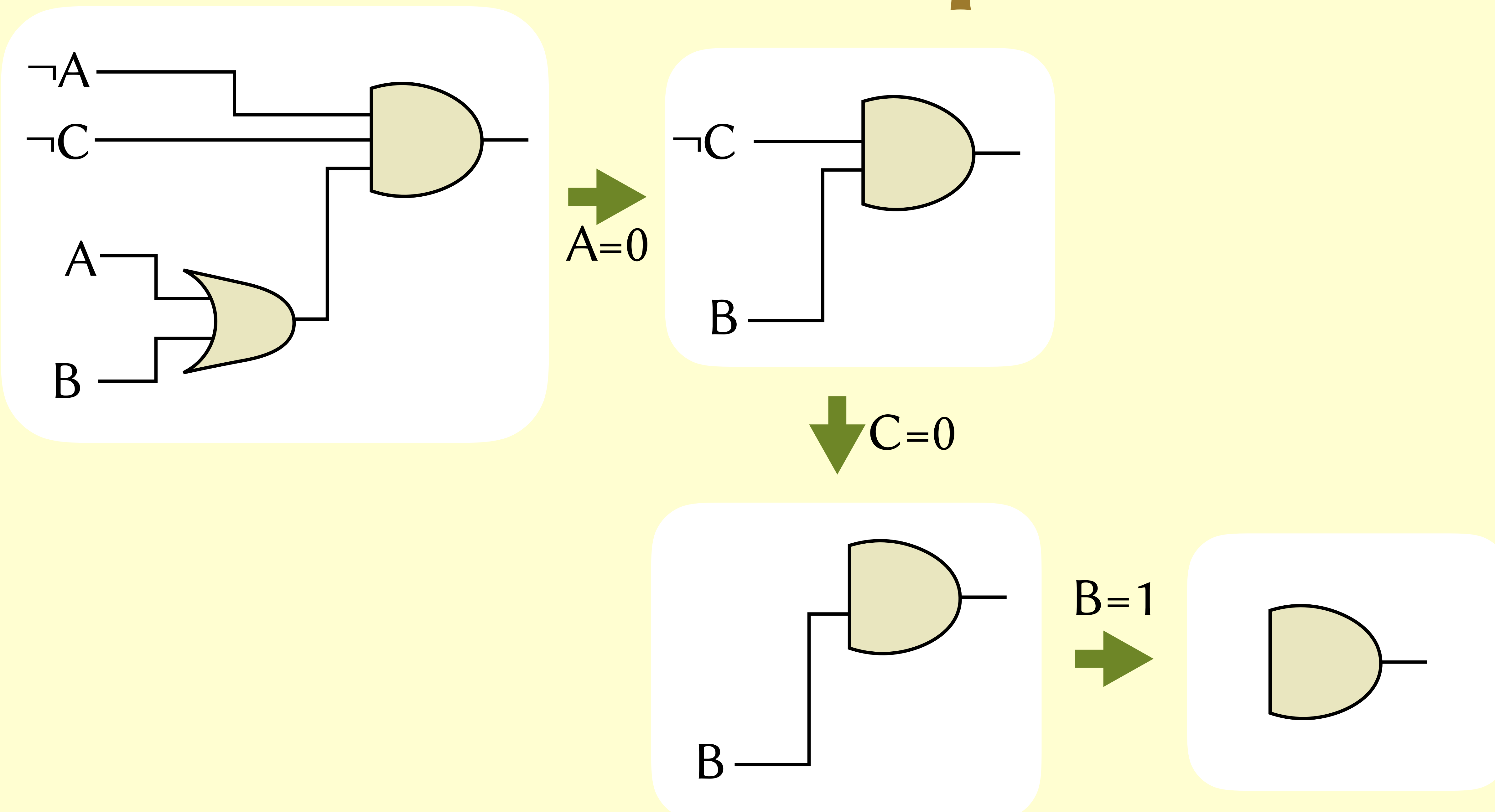

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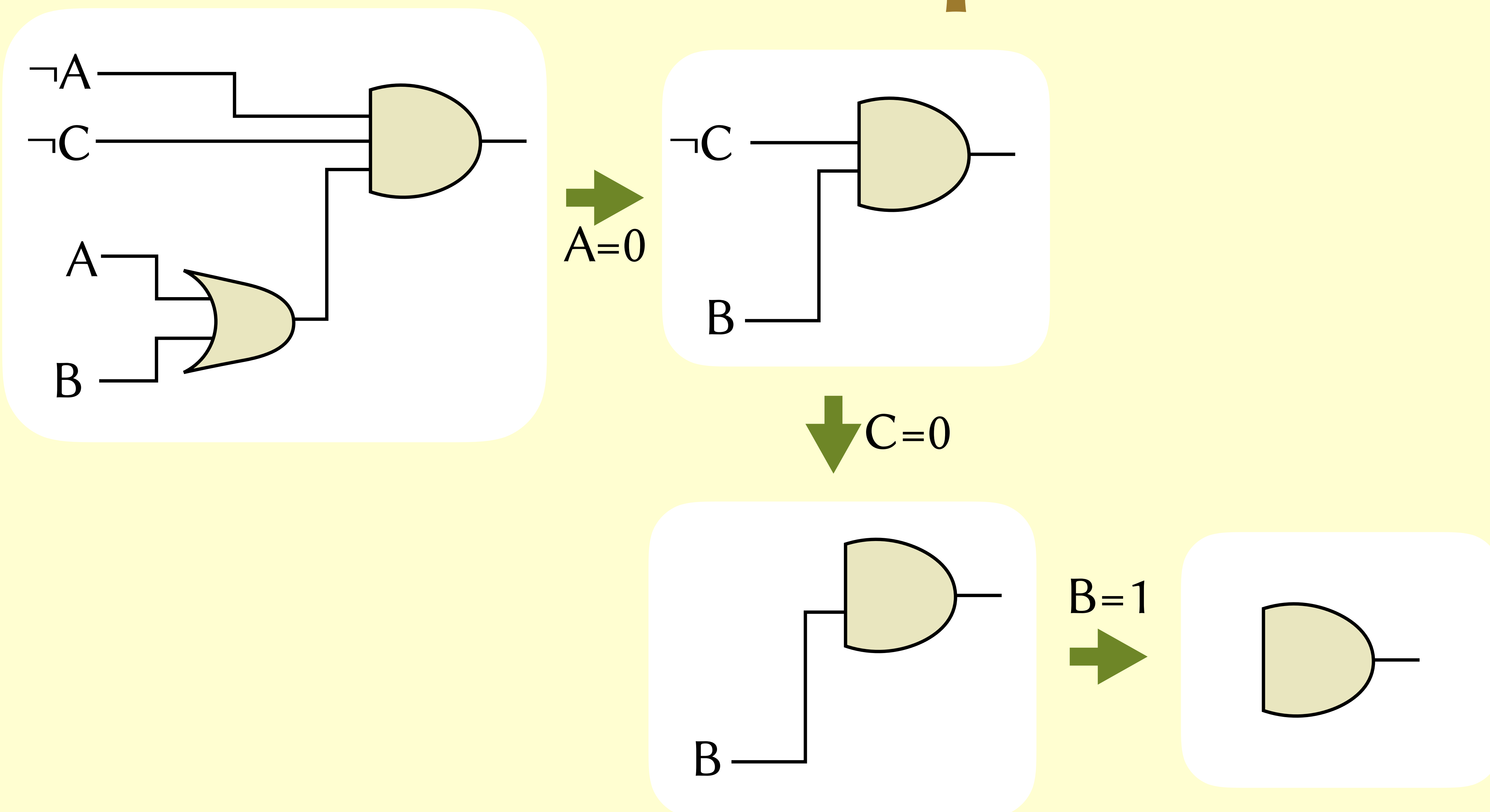

Satisfiable, e.g.
when A=0, B=1, C=0

DP example 2

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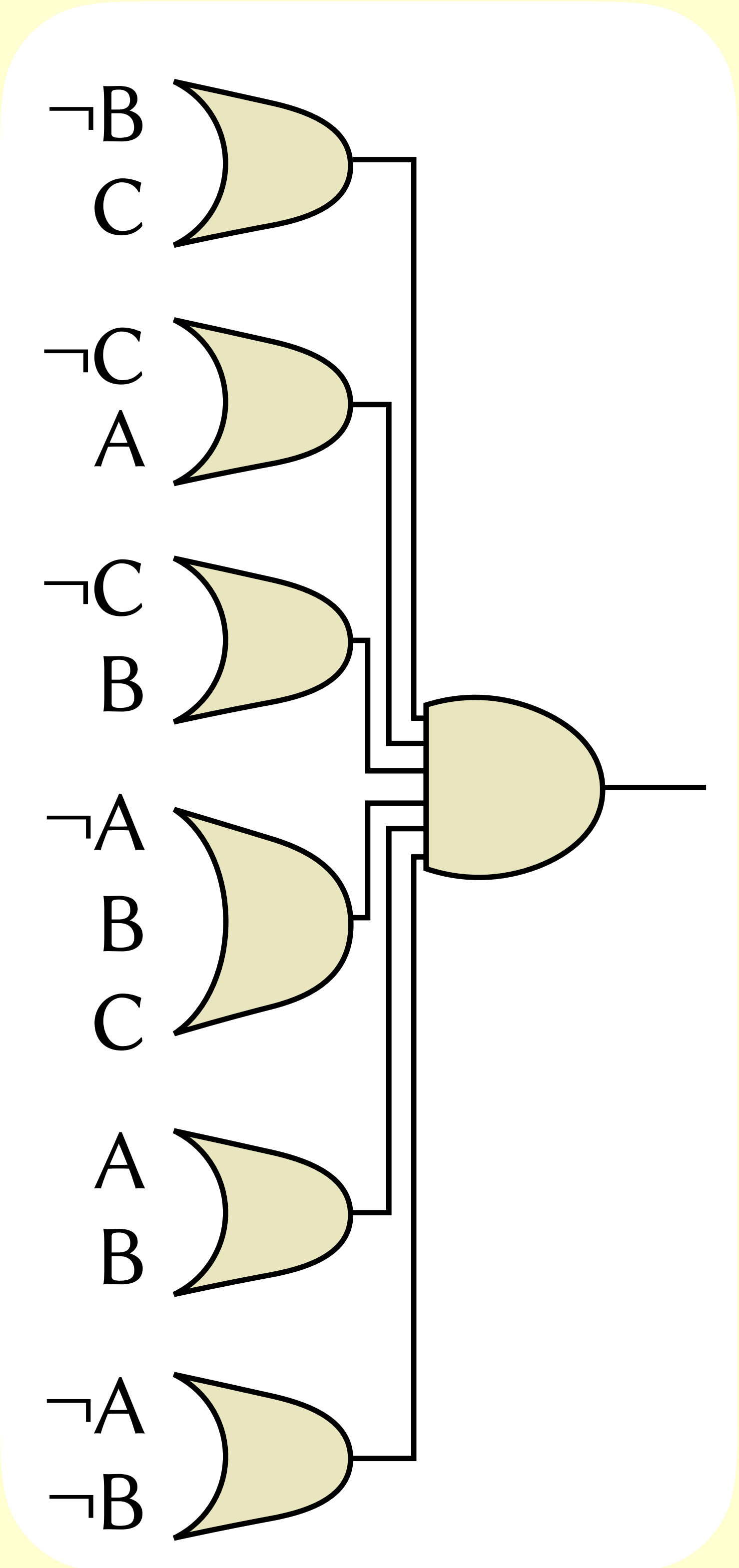

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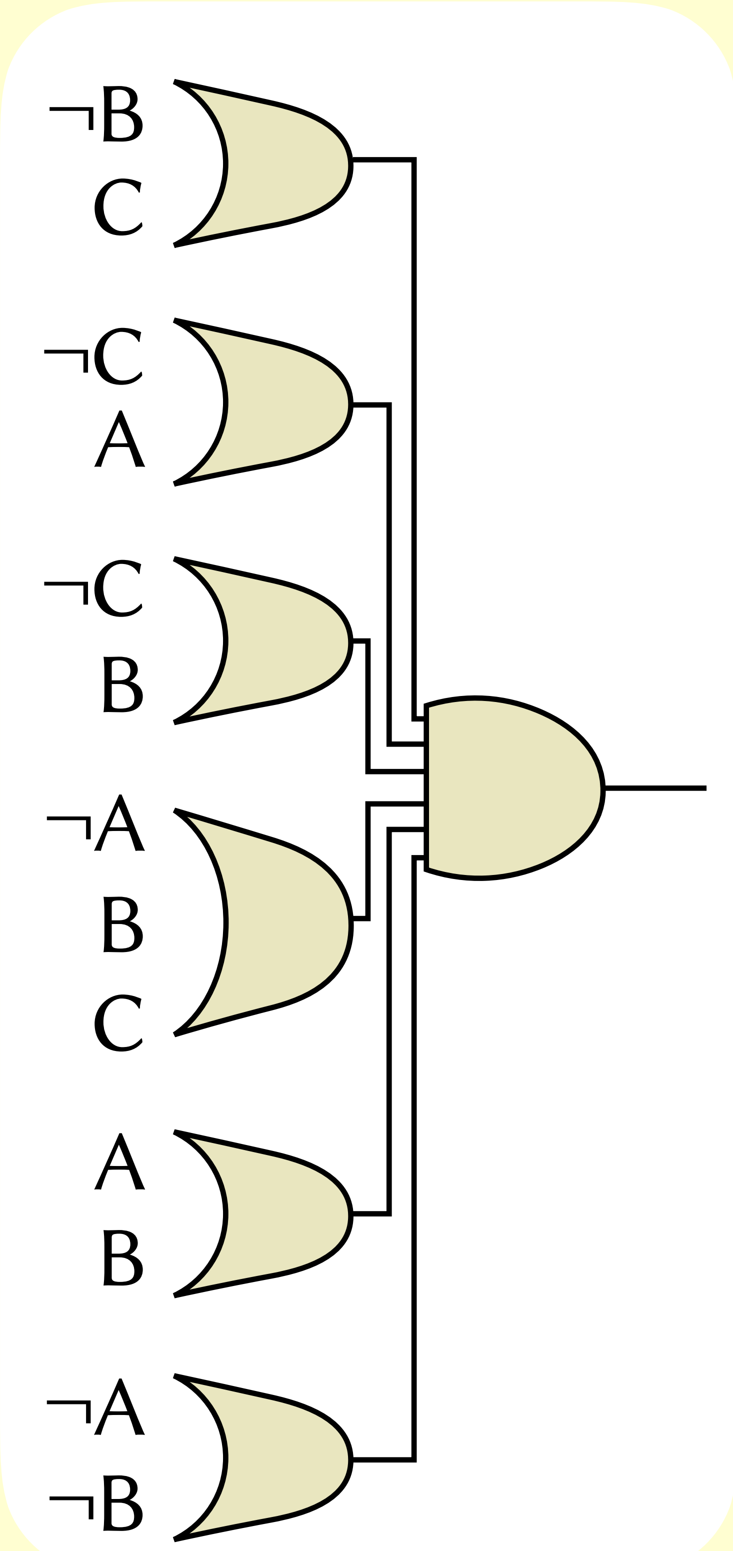
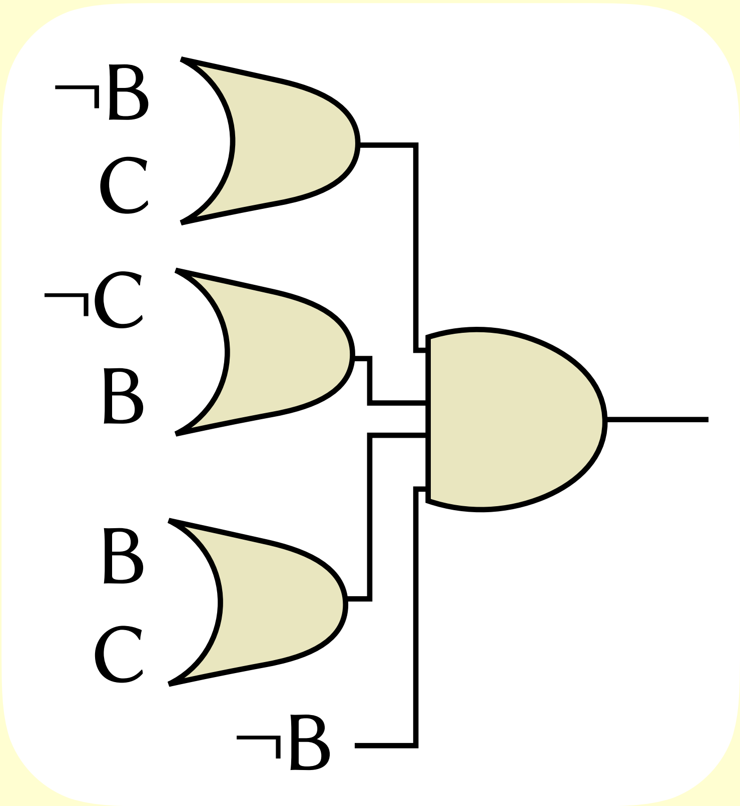


A=1



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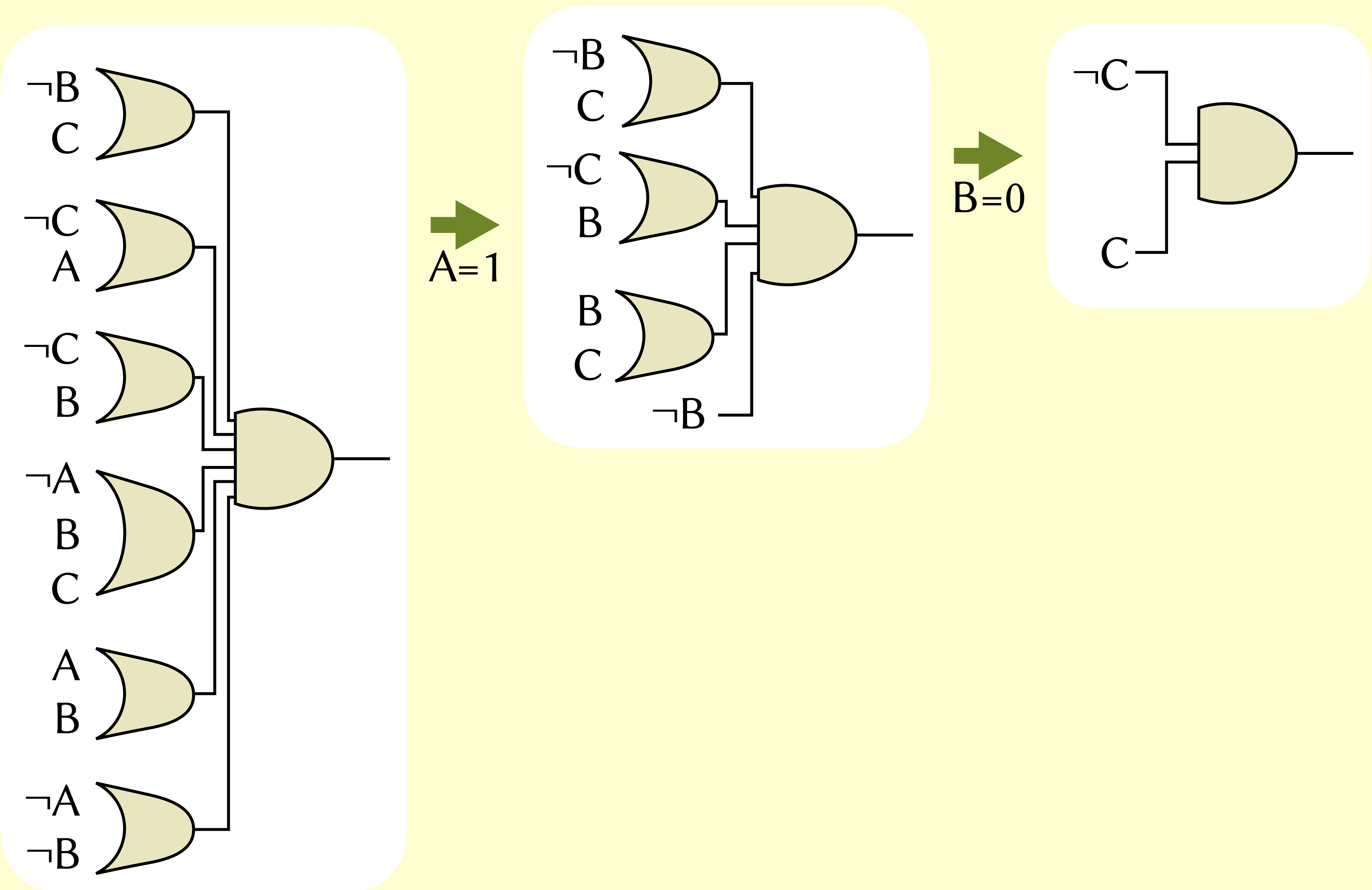

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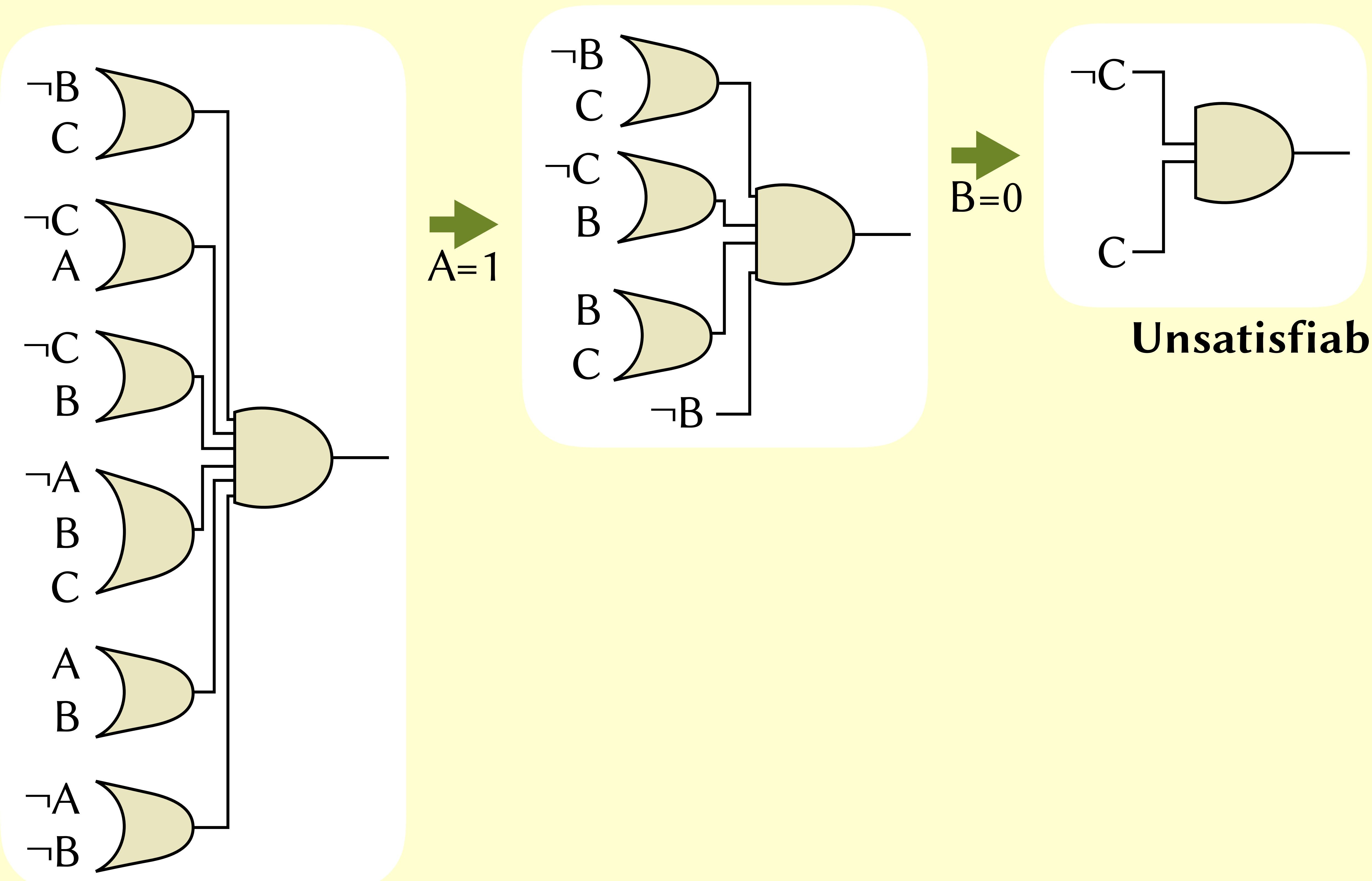

Unsatisfiable

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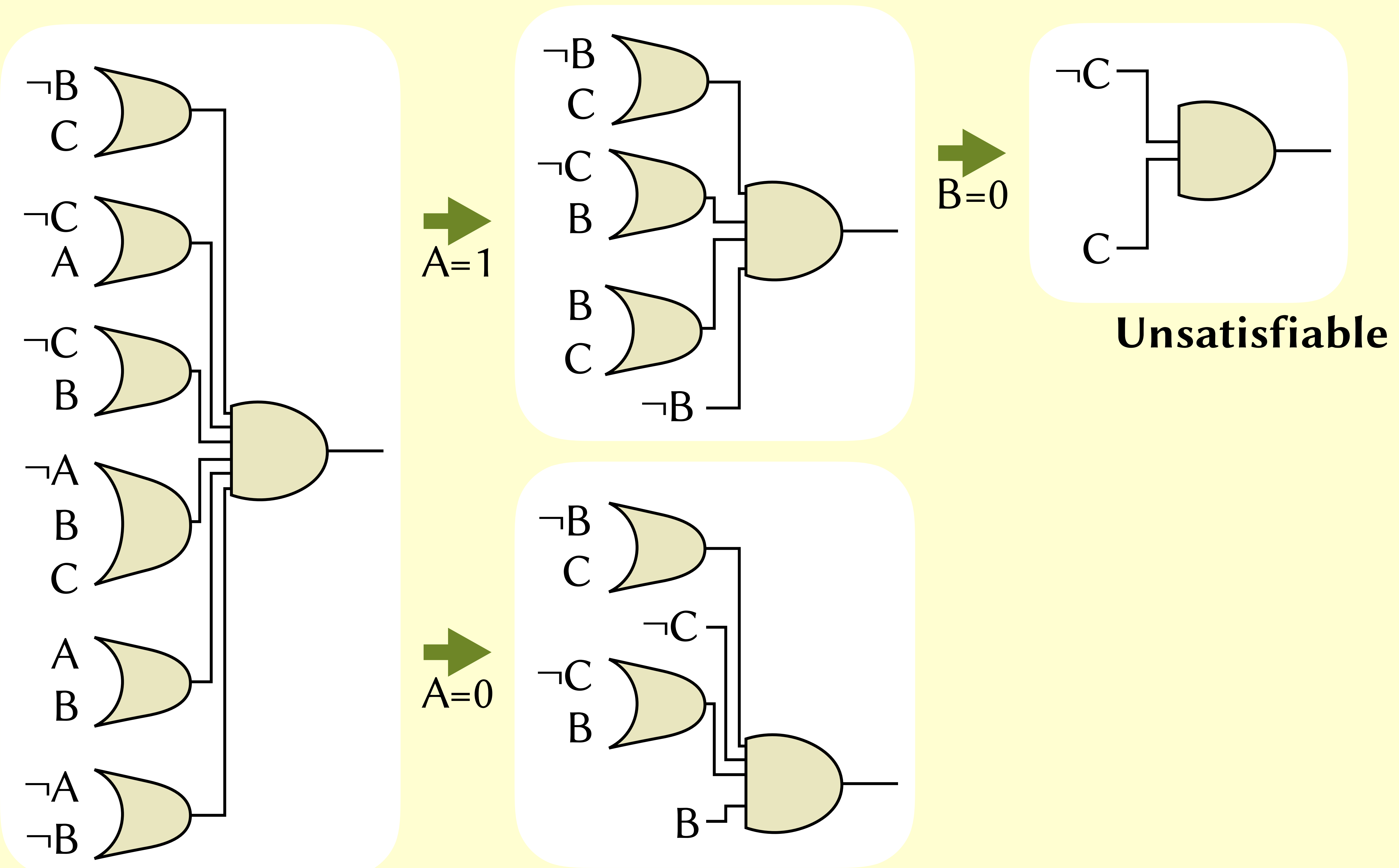

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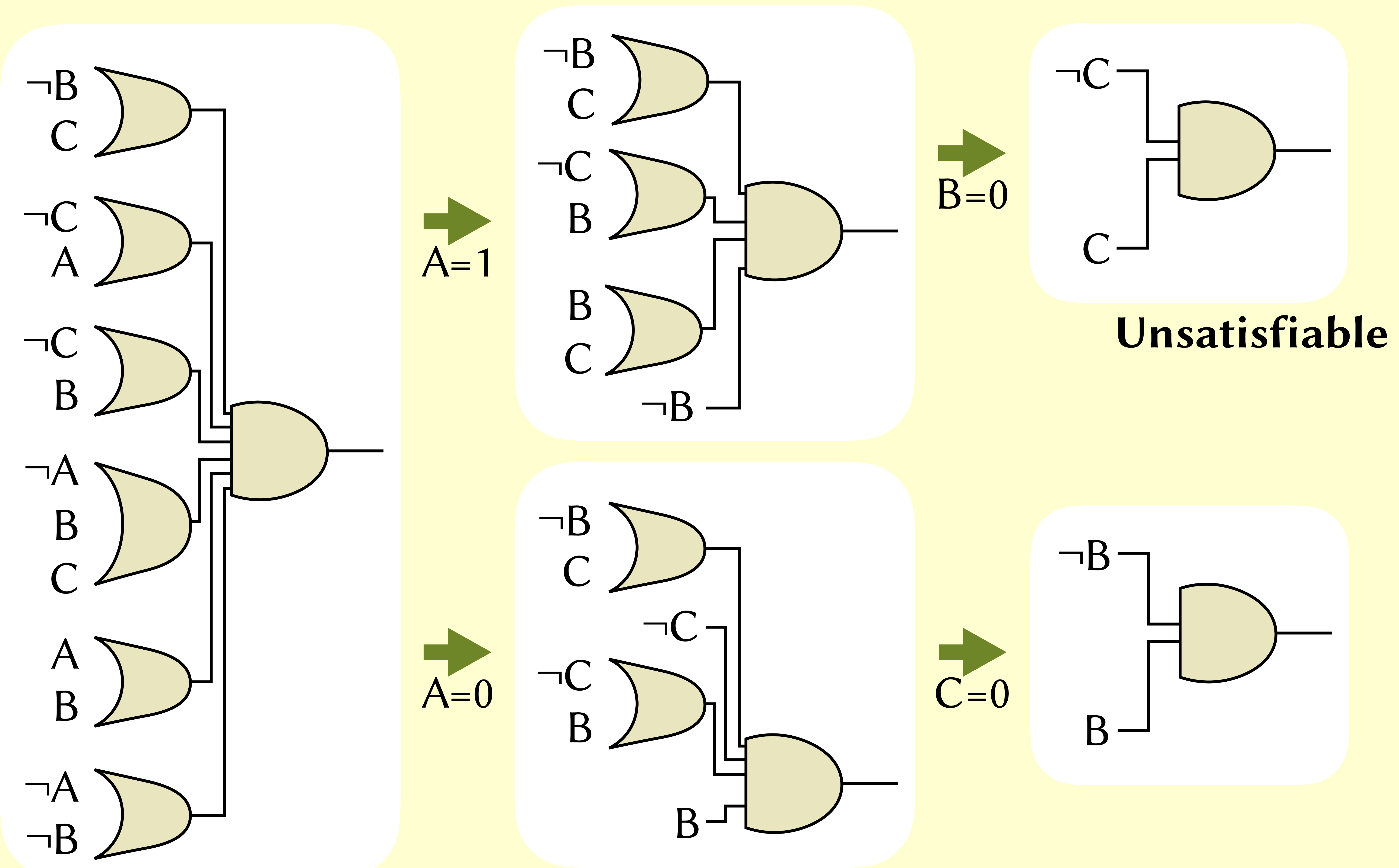
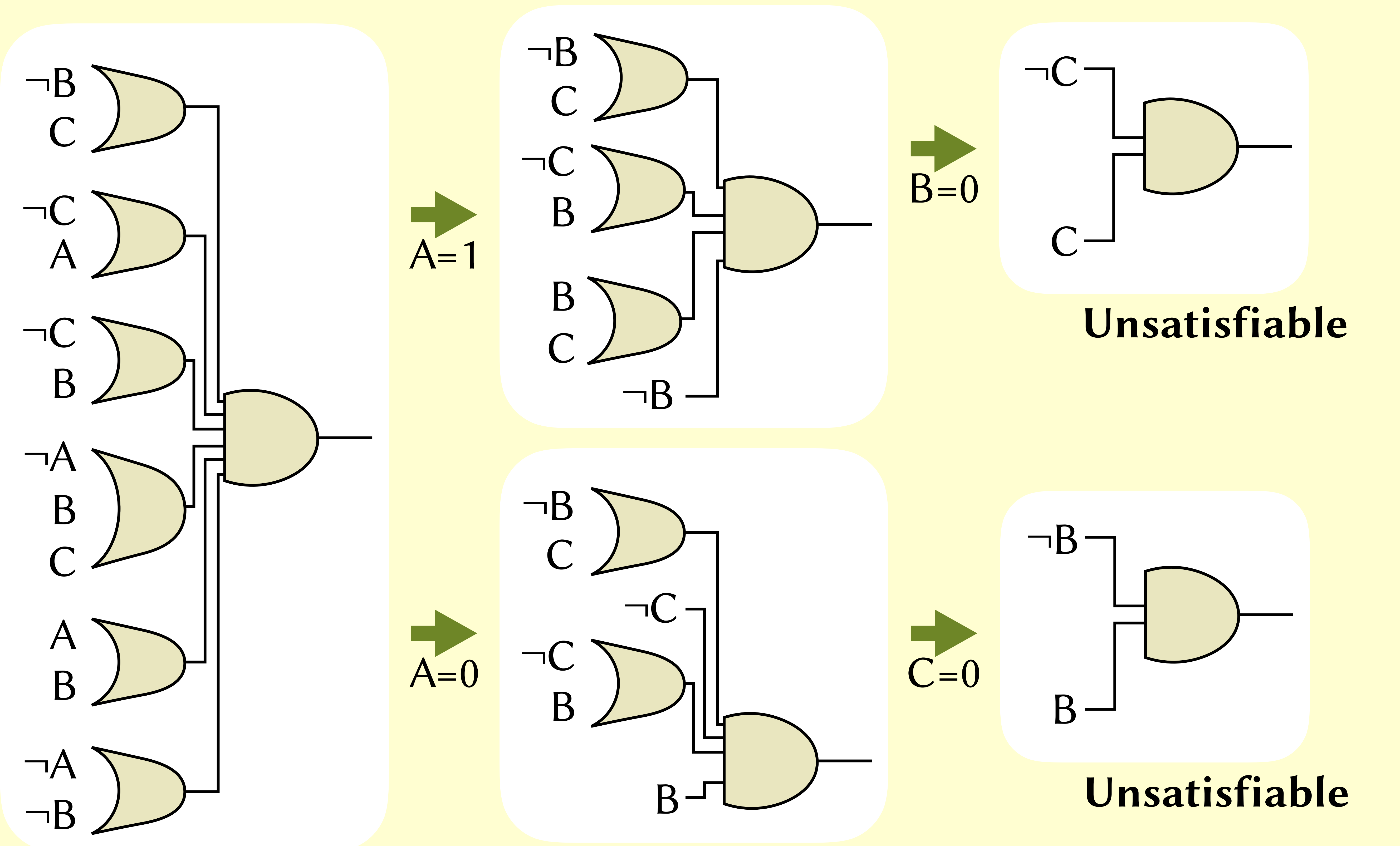


¬B
C
¬C
A
¬C
B
¬A
B
C
A
B
¬A
¬B
A=1
¬B
C
¬C
B
B
C
¬B
B=0
¬C
C
Unsatisfiable
A=0
¬B
C
¬C
¬C
B
B
C=0
¬B
B

DP example 2



¬B
C
¬C
A
¬C
B
¬A
B
C
A
B
¬A
¬B
A=1
A=0
¬B
C
¬C
B
B
C
¬B
¬B
C
¬C
¬C
B
B
B=0
C=0
¬C
C
Unsatisfiable
¬B
B
Unsatisfiable

Towards SMT solving

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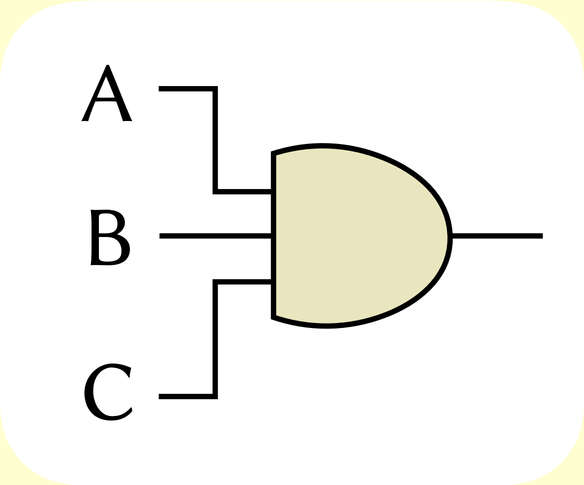

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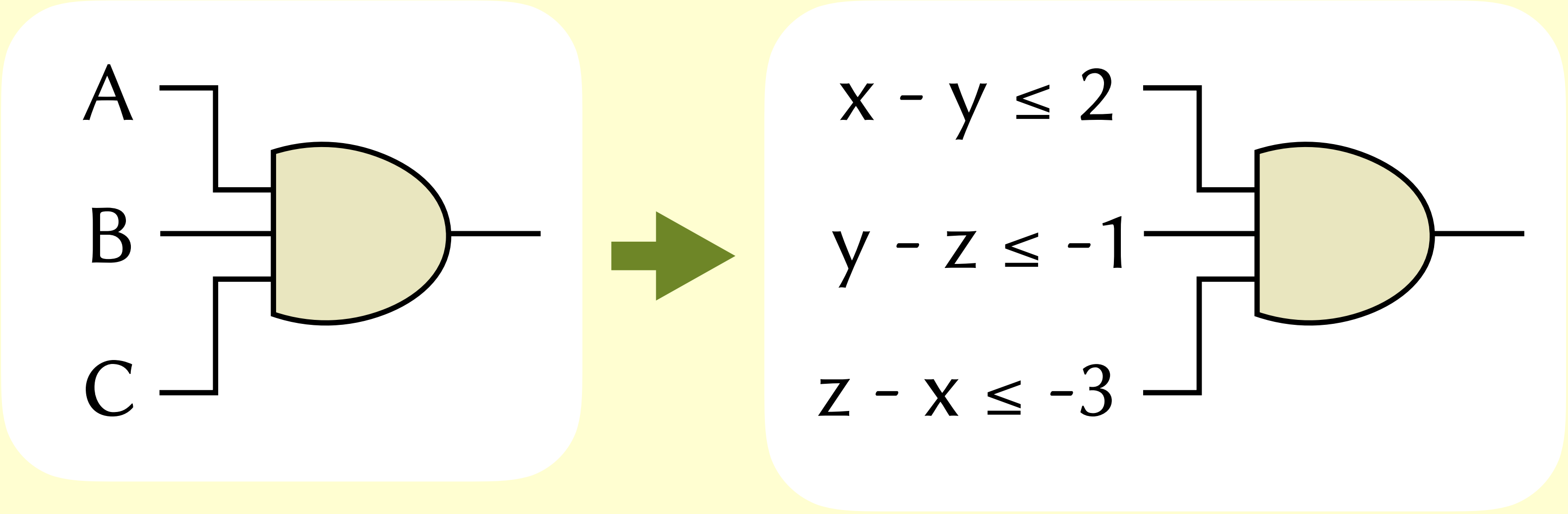

Some theories

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1904–c.1943

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1. $\neg (x + 1 = 0)$
2. $x + 1 = y + 1 \Rightarrow x = y$
3. $x + 0 = x$
4. $x + (y + 1) = (x + y) + 1$
5. $(P(0) \land (\forall x.\ P(x) \Rightarrow P(x+1))) \Rightarrow \forall y.\ P(y)$ (for any P)

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- **Non-linear arithmetic**, which allows queries like:

$$(\sin(x)^3 = \cos(\log(y) \cdot x) \vee b \vee -x^2 \geq 2.3y) \wedge \left(\neg b \vee y < -34.4 \vee \exp(x) > \frac{y}{x}\right)$$

Some theories

- **Equality and uninterpreted functions**, which knows that you can't have x=y and y=z without x=z, and that you can't have x=y without f(x)=f(y).

- **Difference logic**, where statements take the form x - y ≤ c.

- **Presburger arithmetic**, which allows statements about integers containing +, -, 0, 1, and =.

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- **Theory of arrays**, **theory of bit-vectors**, etc.

Decidability of Presburger

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$x + y = z$

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	1	2	4	8	16	32	64
x =	0	1	0	0	1	0	0
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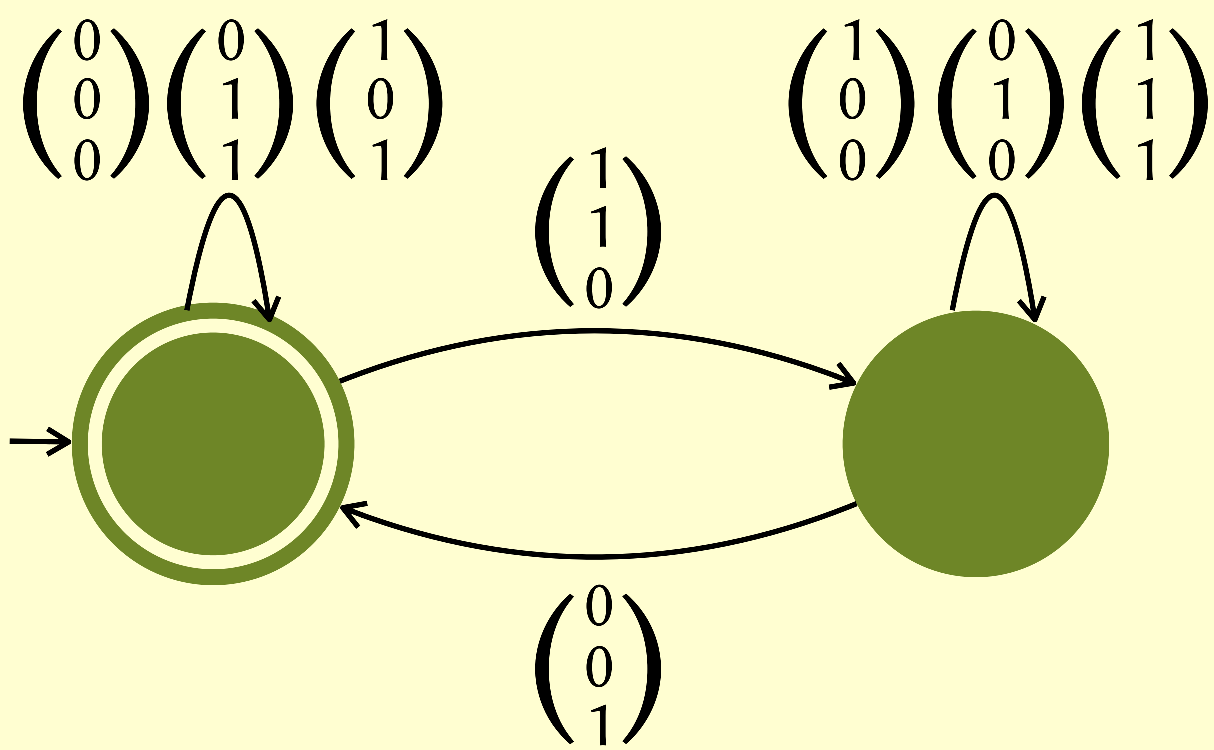

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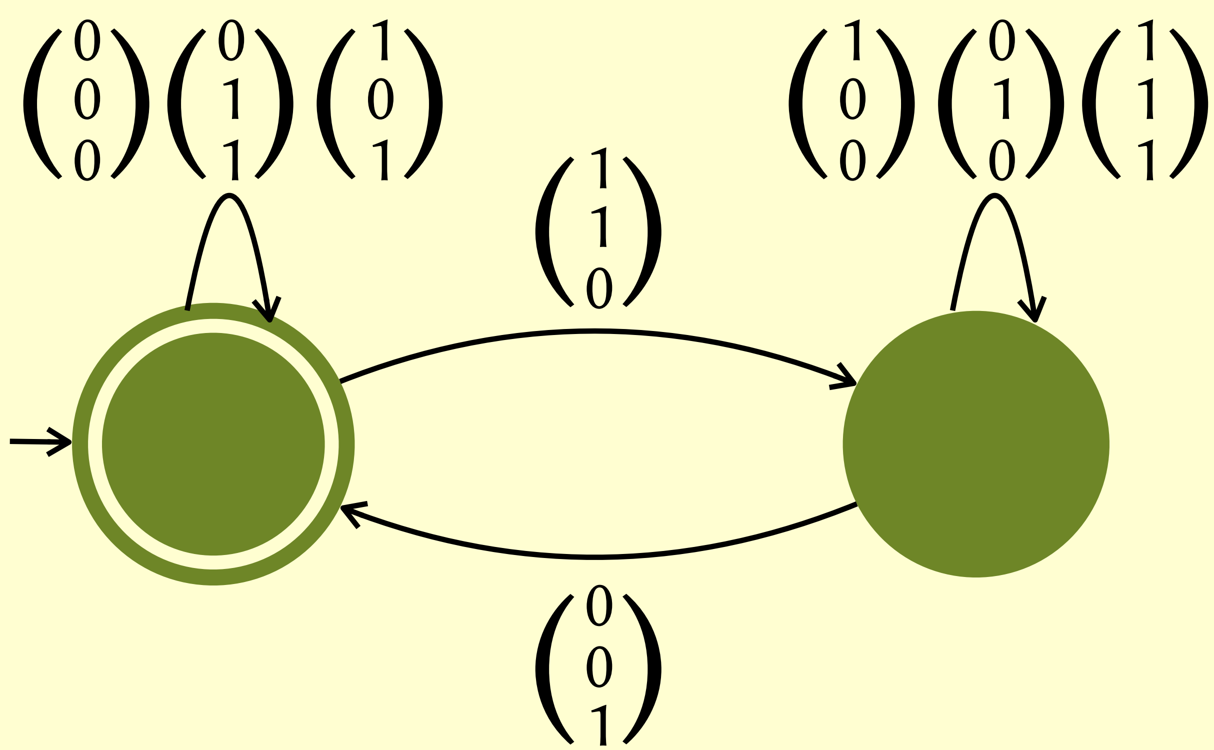

Julius Richard Büchi
1924–1984

Adding multiplication

1. $\neg (x + 1 = 0)$
2. $x + 1 = y + 1 \Longrightarrow x = y$
3. $x + 0 = x$
4. $x + (y + 1) = (x + y) + 1$
5. $x \times 0 = 0$
6. $x \times (y + 1) = x \times y + x$
7. $(P(0) \wedge (\forall x.\ P(x) \Longrightarrow P(x+1))) \Longrightarrow \forall y.\ P(y)$ (for any P)

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Giuseppe Peano
1858–1932

Undecidability of Peano

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- Now we have multiplication, we can write a statement representing the Collatz conjecture: does there exist an infinite sequence of positive integers x_0, x_1, x_2, ... such that

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\begin{array}{ll}
2 \times x_{i+1} = x_i & \text{if } x_i \text{ is even} \\
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- So **if** arithmetic with multiplication were decidable, we could solve the Collatz conjecture automatically!

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- The Halting Problem can be encoded as a statement about arithmetic.
- So I can use my algorithm to solve the Halting Problem.
- But it is impossible to write an algorithm to solve the Halting Problem!
- So it must also be impossible to write an algorithm to decide whether arithmetic statements are true or not.

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- **Task.** Write a program `halts` with the following declaration:

  ```
  int halts(char *P, char *D);
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```

`halts(S6, _)` = 1

Aside: Halting Problem

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```
S1

S2

S3

S4

S5

S6
```

Aside: Halting Problem

	S1	S2	S3	S4	S5	S6
S1						
S2						
S3						
S4						
S5						
S6						

Aside: Halting Problem

	S1	S2	S3	S4	S5	S6
S1						
S2						
S3						
S4						
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S6						

strings interpreted as source code

Aside: Halting Problem

input strings

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	S1	S2	S3	S4	S5	S6
S1						
S2						
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S4						
S5						
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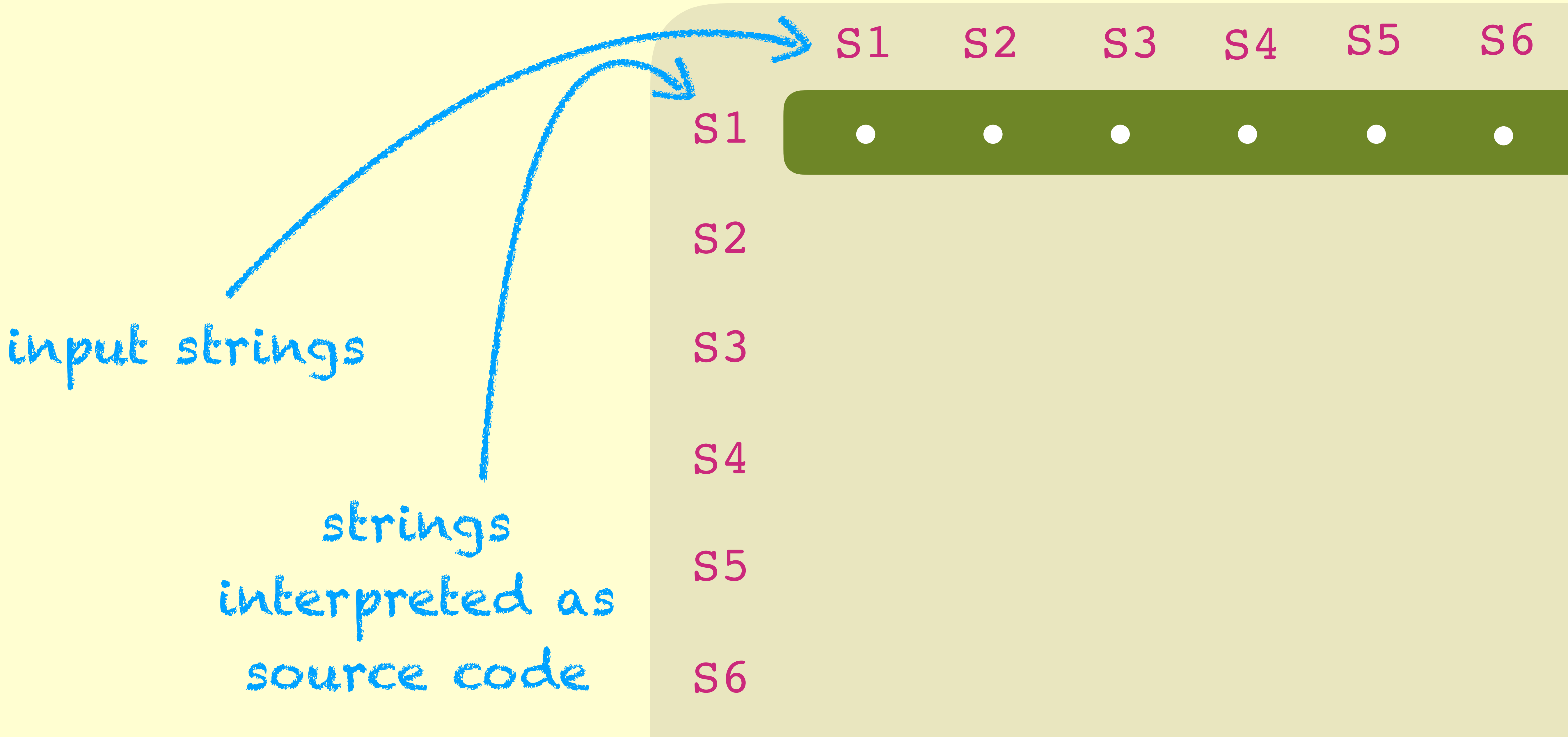

Aside: Halting Problem

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	S1	S2	S3	S4	S5	S6
S1	•	•	•	•	•	•
S2						
S3						
S4						
S5						
S6	✔	✔	✔	✔	✔	✔

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	S1	S2	S3	S4	S5	S6
S1	•	•	•	•	•	•
S2	✔	✔	✔	•	•	•
S3	•	✔	•	✔	•	•
S4	•	•	•	•	•	✔
S5	✔	•	✔	•	✔	•
S6	✔	✔	✔	✔	✔	✔

Aside: Halting Problem

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S1	•	•	•	•	•	•
S2	✔	✔	✔	•	•	•
S3	•	✔	•	✔	•	•
S4	•	•	•	•	•	✔
S5	✔	•	✔	•	✔	•
S6	✔	✔	✔	✔	✔	✔

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•	✔	•	•	✔	✔

	S1	S2	S3	S4	S5	S6
S1	•	•	•	•	•	•
S2	✔	✔	✔	•	•	•
S3	•	✔	•	✔	•	•
S4	•	•	•	•	•	✔
S5	✔	•	✔	•	✔	•
S6	✔	✔	✔	✔	✔	✔

input strings

strings
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✔	•	✔	✔	•	•

	S1	S2	S3	S4	S5	S6
S1	•	•	•	•	•	•
S2	✔	✔	✔	•	•	•
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S5	✔	•	✔	•	✔	•
S6	✔	✔	✔	✔	✔	✔

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S1	•	•	•	•	•	•
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S =
"int s(char *D) {
   if (halts(D, D))
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 }"
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S	✔	•	✔	✔	•	•

	S1	S2	S3	S4	S5	S6
S1	•	•	•	•	•	•
S2	✔	✔	✔	•	•	•
S3	•	✔	•	✔	•	•
S4	•	•	•	•	•	✔
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$\Big\downarrow$ defn of `halts`

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{} & \texttt{s(S)}\text{ doesn't halt} \arrow[l, "\text{defn of }\texttt{s}"']
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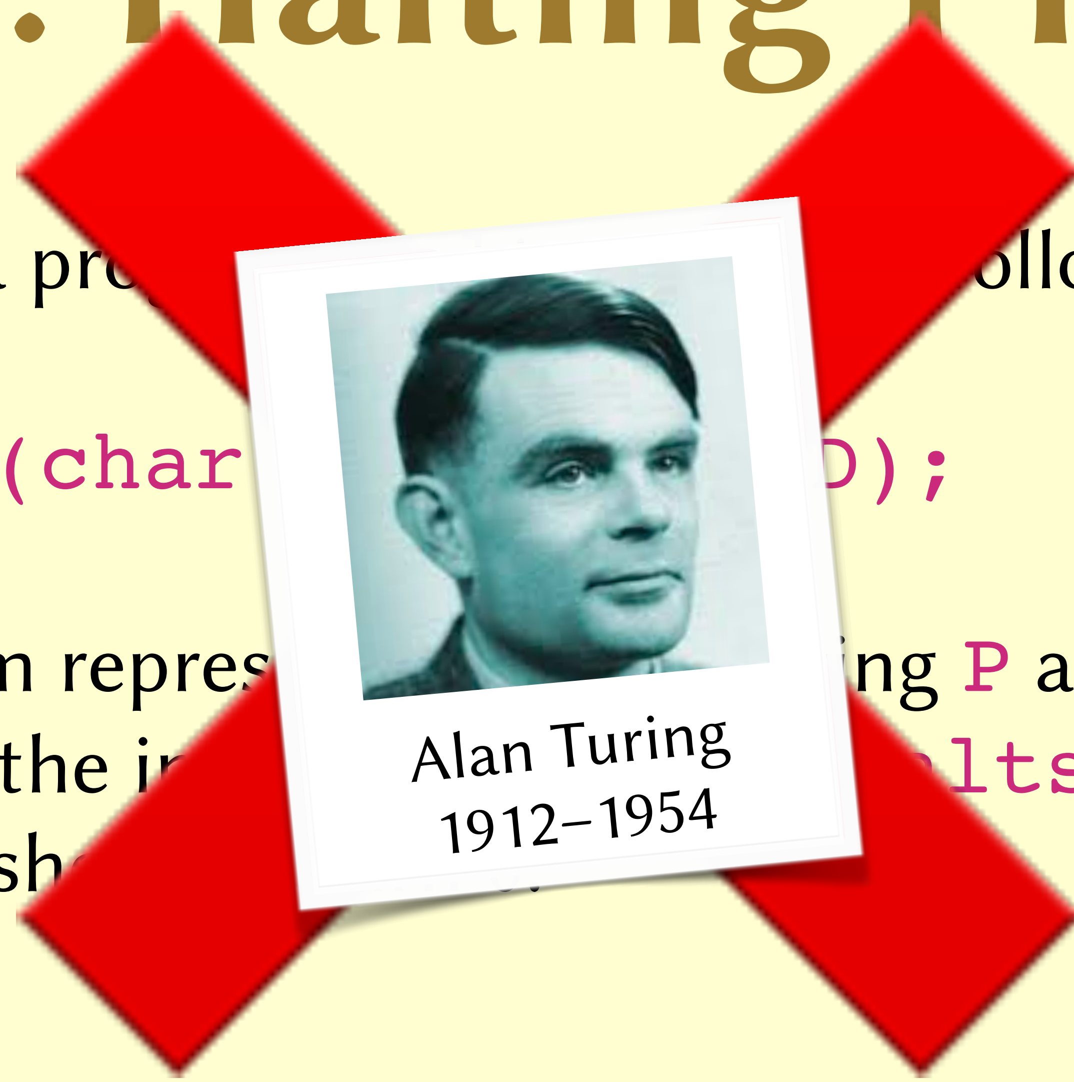

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- How do these automatic provers work?